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Slide of the Seminar

Optimum transport and exact coherent states: the Rayleigh-Bérnard example

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Optimum transport and exact coherent states: the Rayleigh-Bénard example

Fabian Waleffe

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Department of Engineering Physics

with
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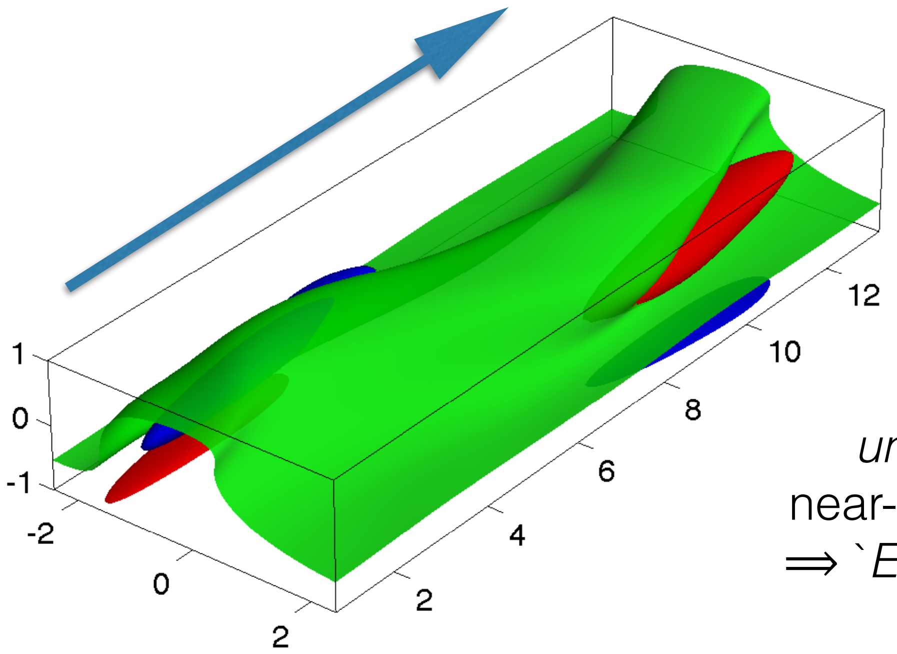
UNIVERSITY OF WISCONSIN–MADISON

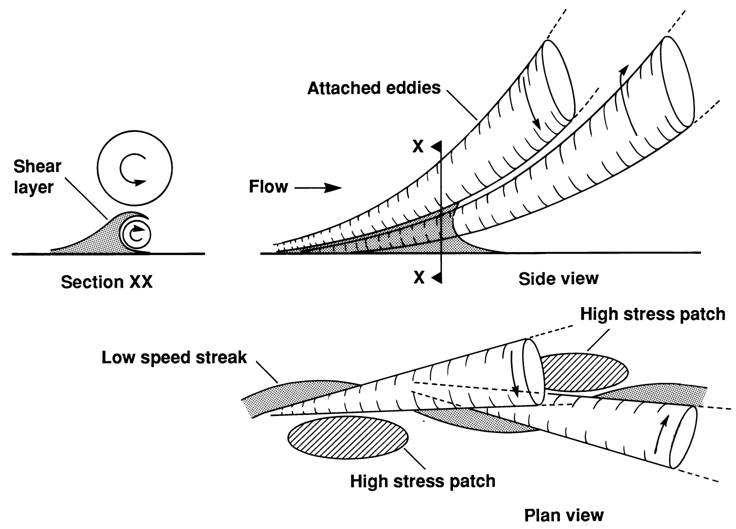
Unstable coherent states in shear flows



3D Traveling Wave in Plane Poiseuille flow

unstable, yet captures near-wall structure quite well $\Longrightarrow$ `*Exact coherent structure*'

Derek Stretch, CTR 1990
Structure of high drag regions in turbulent channel flow
(KMM $R_\tau = 180$)

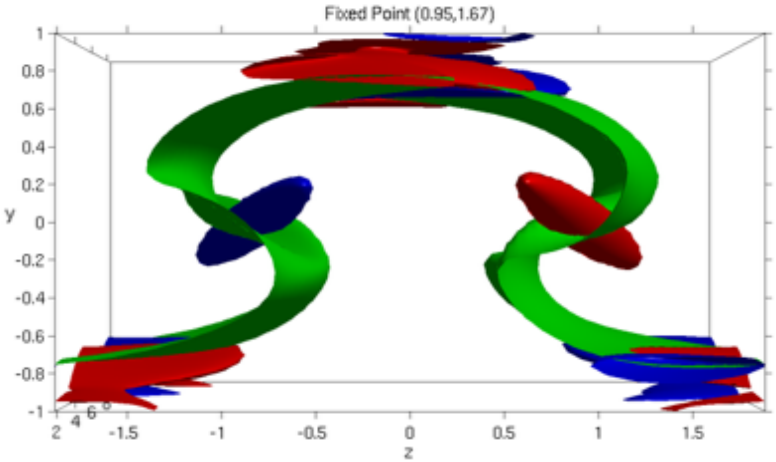


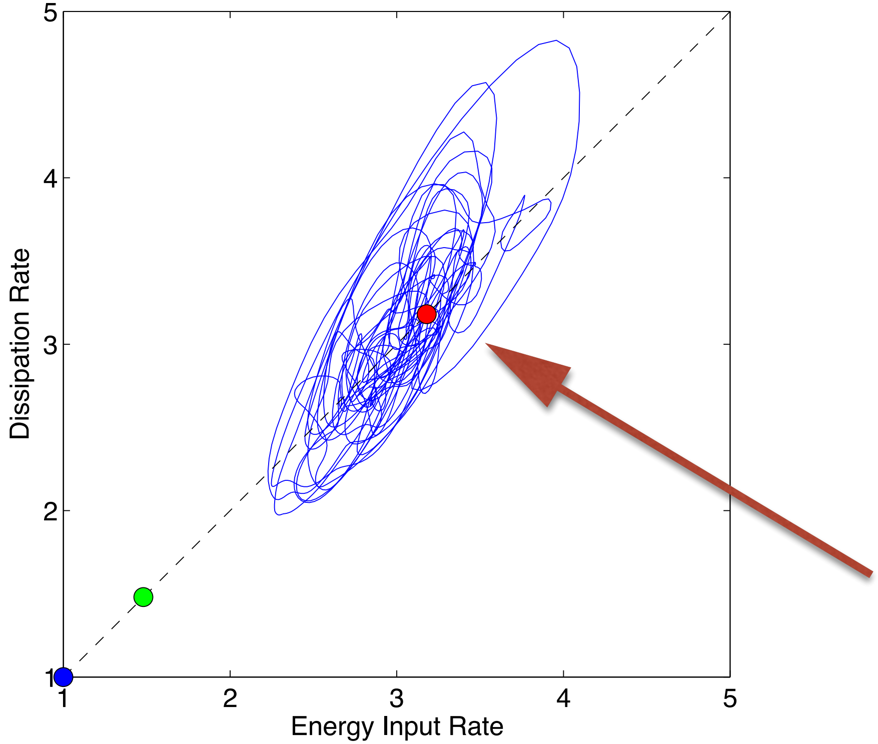
`Exact' Coherent structures

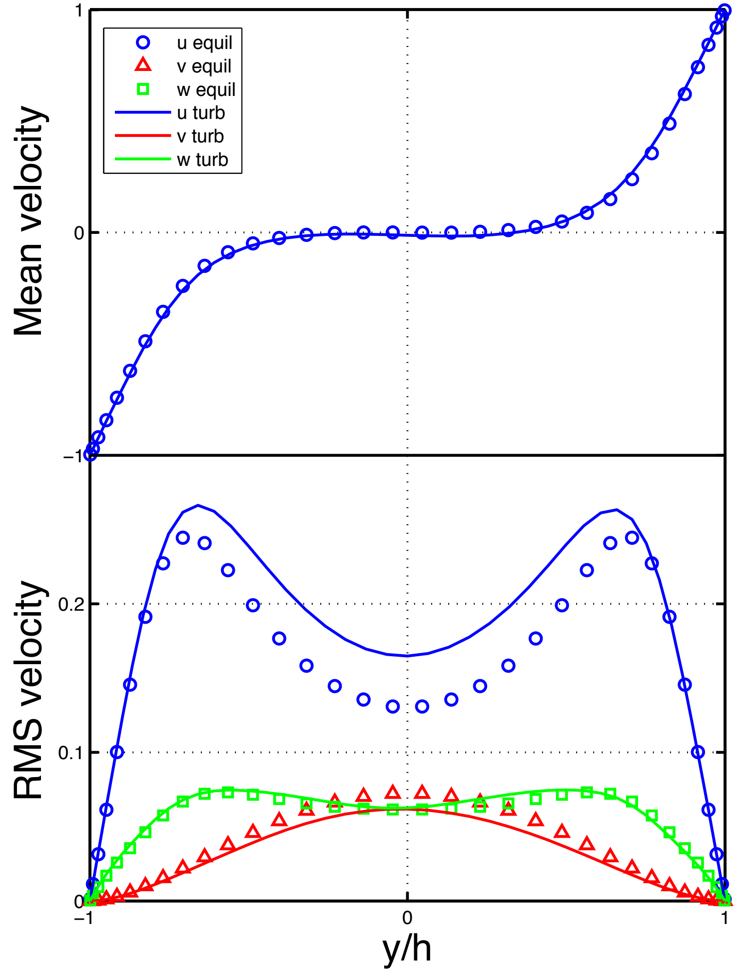
self-consistently combine:

- streaks
- staggered quasi streamwise vortices
- `sweeps' (Q4) and `ejections' (Q2)
- solve Navier-Stokes!   (hence `exact')

Unstable 3D Steady State in Plane Couette Flow

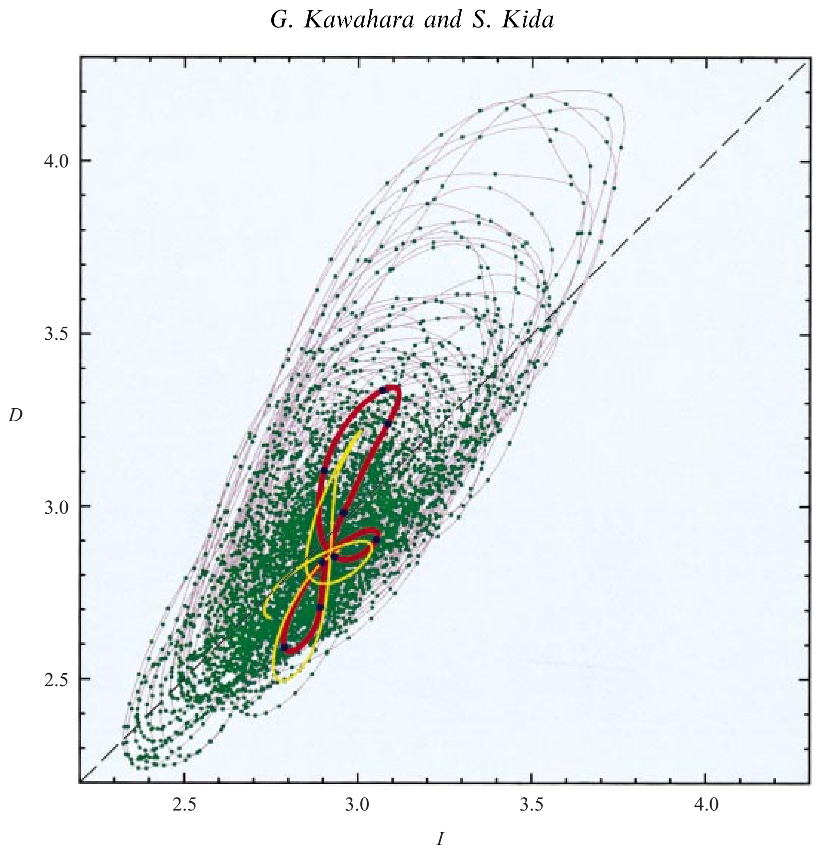


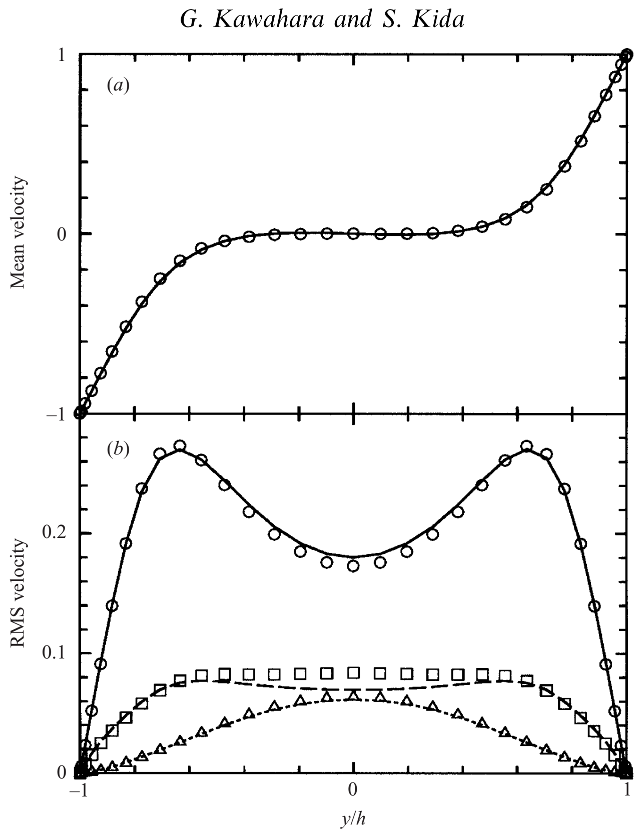




One unstable steady state (upper branch)
captures statistics quite well

Unstable time-periodic solutions in Plane Couette Flow





Mean U

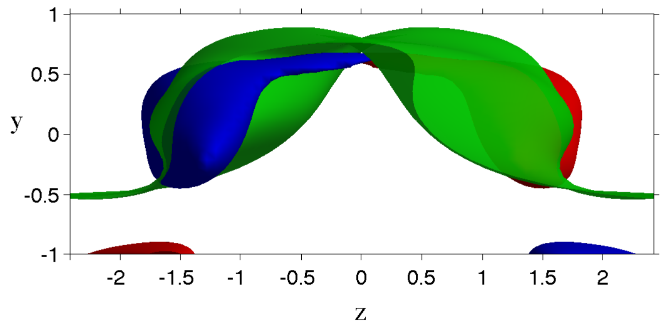
u, v, w
rms

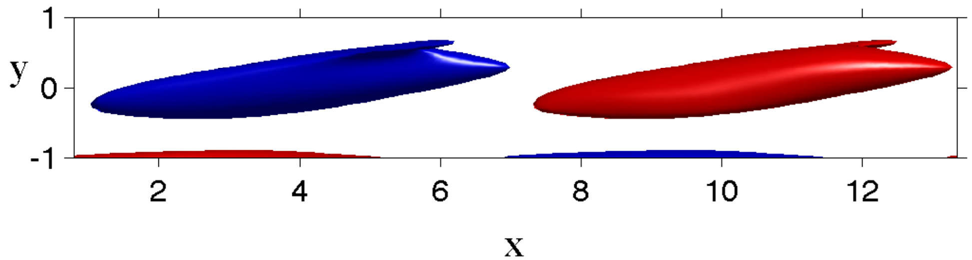
Energy Input & Dissipation

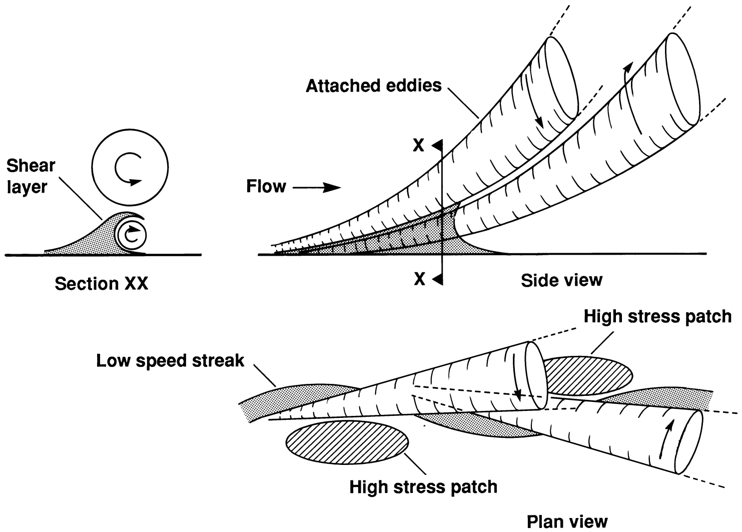
One unstable periodic state captures *more* statistics *better*

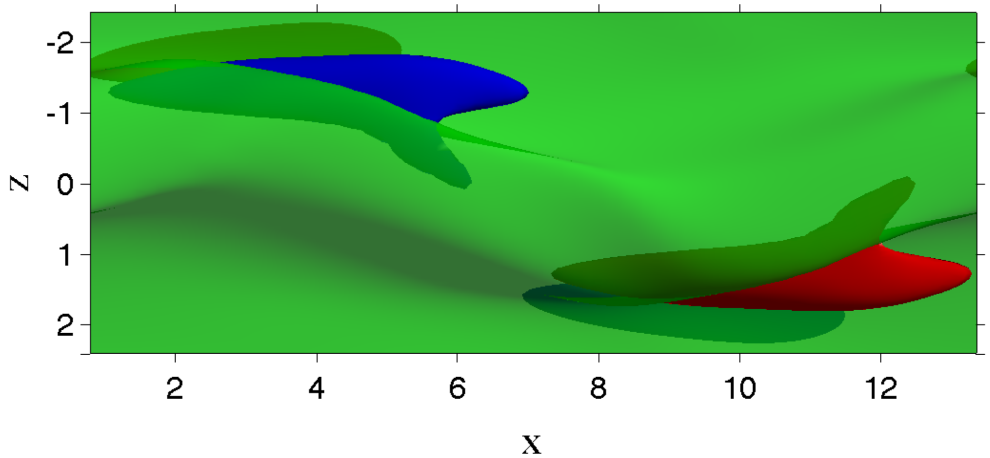
Kawahara & Kida, JFM 2001

`Optimum' channel flow ECS



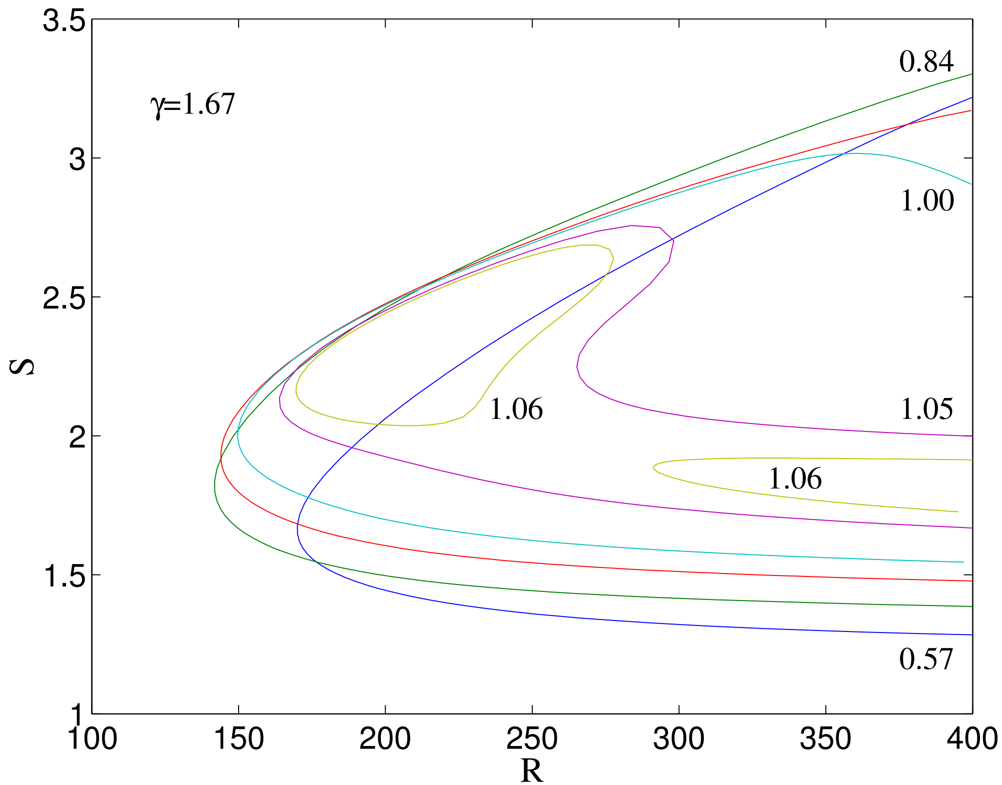


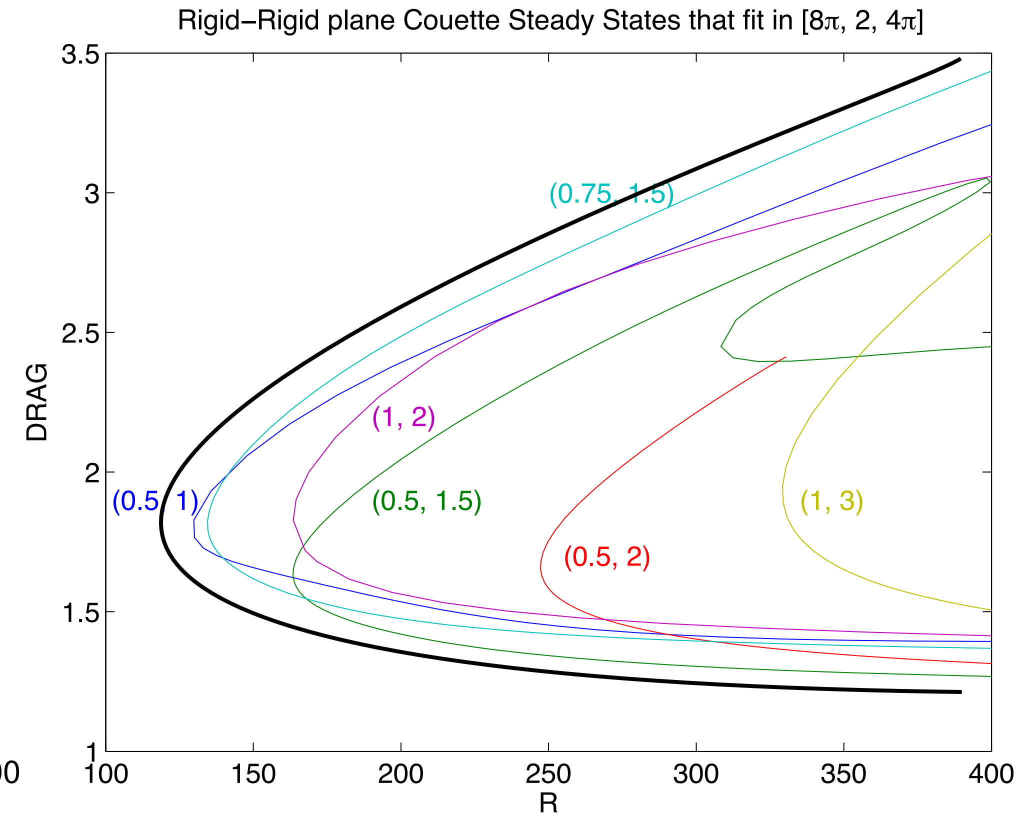




$$\min R_\tau = 2h^+ = 44 \ \text{ for } L_x^+ = 274,\ L_z^+ = 105$$

Lots of EQs, TWs, POs: which ones matter?





Attempted to compute **envelope**,
largest and smallest shear solutions
optimizing over both horizontal wavenumbers
(Jue Wang & FW, 2003, *abandoned*)

Computing envelope was too hard

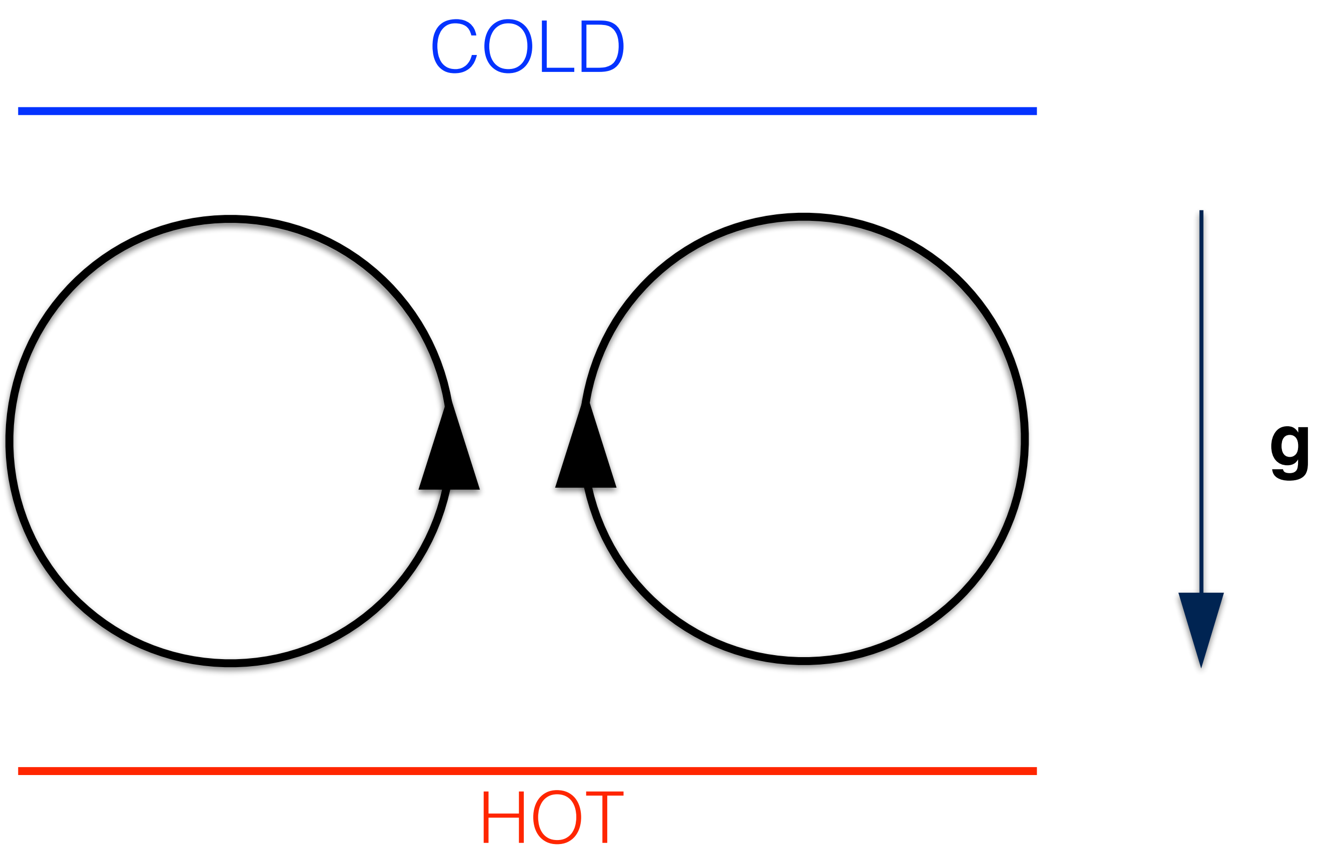
(back then in 2003)

Shear flows are difficult:

ECS are 3D traveling waves, periodic orbits,…

Lots of different solutions

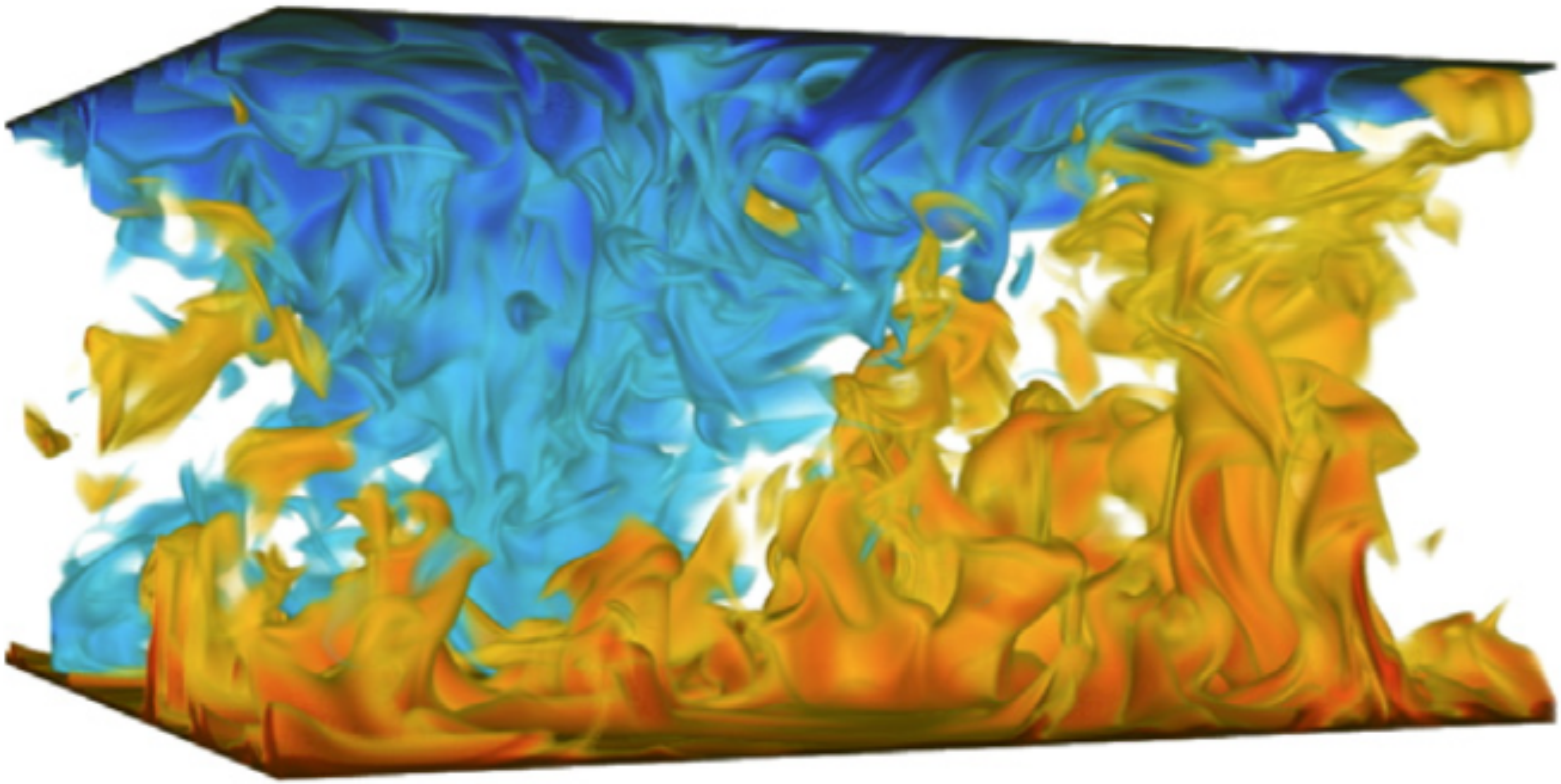
The simpler Rayleigh-Bénard example



Rayleigh-Bénard:
Paradigmatic problem in nonlinear physics

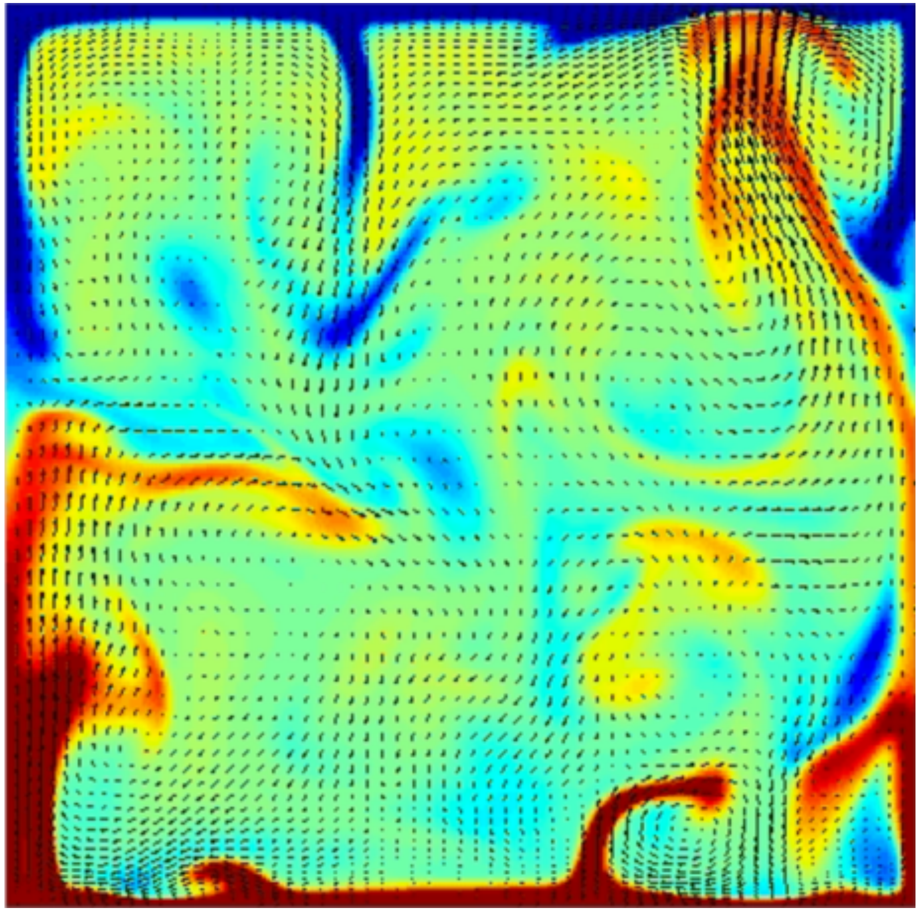
- Fluid instabilities, bifurcations,
- Lorenz eqns, attractor, chaos, …
- Pattern formation, …
- Turbulence…

Turbulence in Rayleigh-Bénard convection



van der Poel *et al.* Computers in Fluids, 2015 $Ra = 10^8, \ Pr = 0.7$

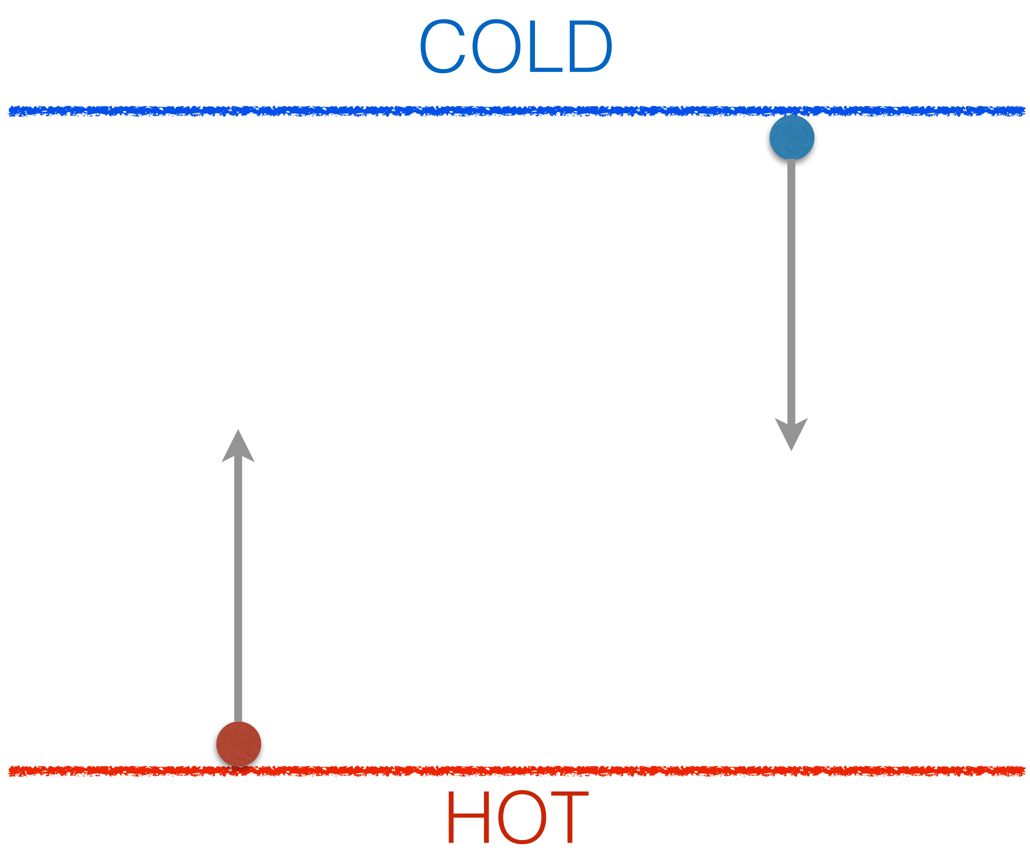
Turbulence in Rayleigh-Bénard convection

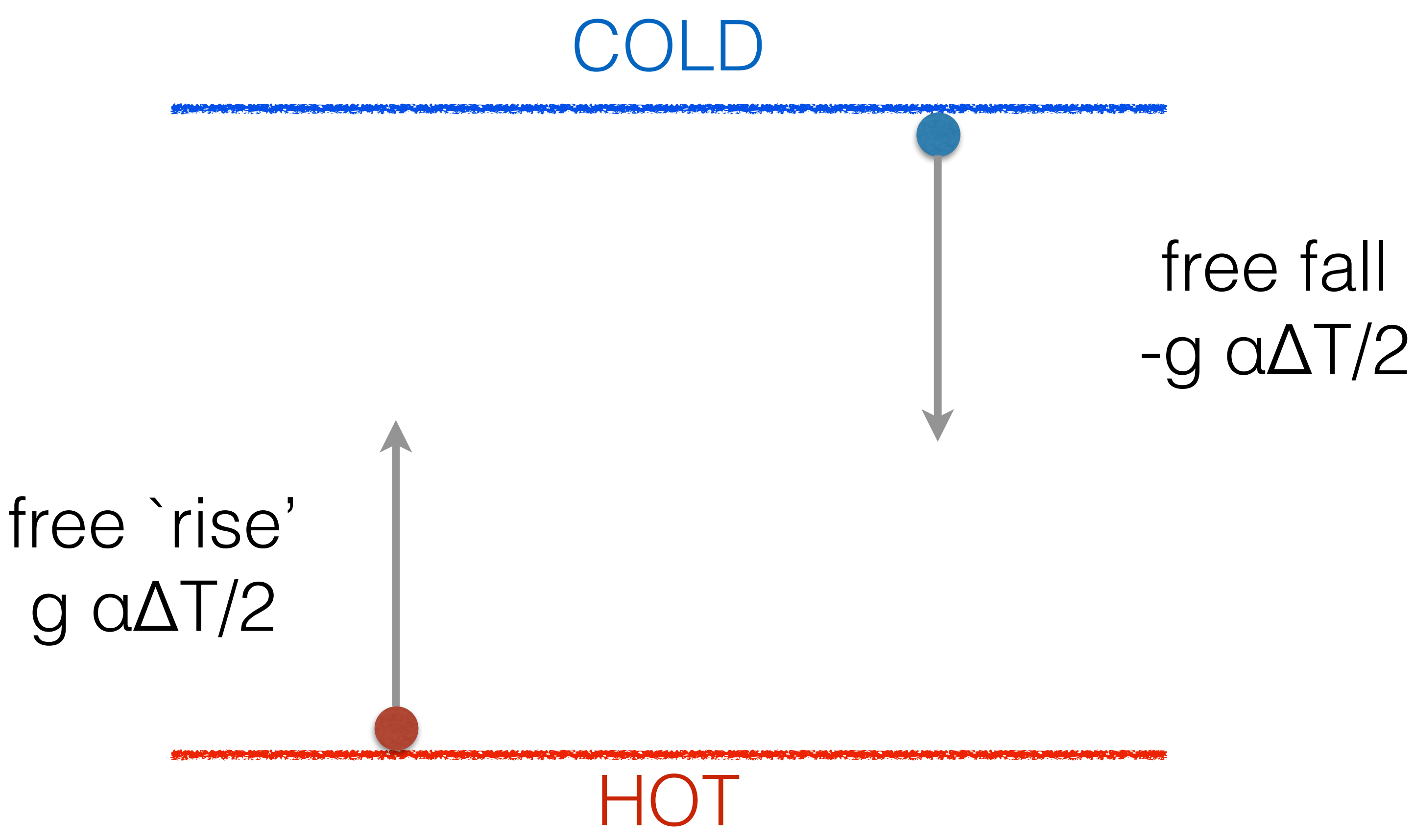


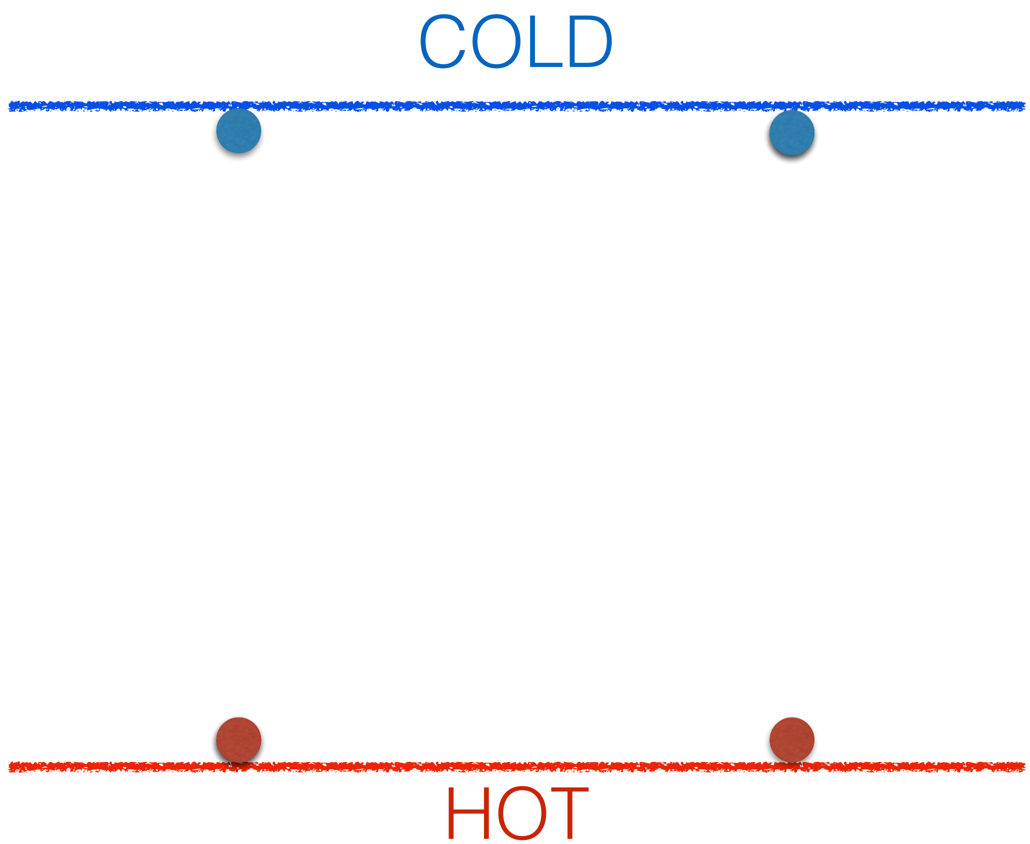
Richard Stevens (cylinder) $Ra = 10^8, \ Pr = 6.4$

Basic question: how much heat is transported for a given ΔT?

Mechanistic picture for inertial scaling




COLD
free fall
-g αΔT/2
free `rise'
g αΔT/2
HOT


COLD
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Mechanistic argument for inertial scaling

- interior at $T = 0$, top wall at $-\Delta T/2$, bottom wall at $\Delta T/2$

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$$Nu = 1 + \frac{\langle VT\rangle}{\kappa\Delta T/H} \sim \frac{1}{4}(Ra\,Pr)^{1/2}$$

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- flux independent of ν, κ. Inertial scaling.
- More sophisticated arguments by Kraichnan, Spiegel, . . . , Grossmann & Lohse. Richer phenomenology, log corrections, various regimes in (Ra, Pr) plane.

Mathematical model: Boussinesq equations

Velocity $\boldsymbol{u}(\boldsymbol{x}, t)$, Temperature $T(\boldsymbol{x}, t)$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p = \nu \nabla^2 \boldsymbol{u} + T \,\hat{\boldsymbol{y}}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \kappa \nabla^2 T$$

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(small) parameters

$$\nu = 4\sqrt{\frac{Pr}{Ra}}, \qquad \kappa = 4\sqrt{\frac{1}{Ra\,Pr}}$$

i.e.

$$Ra = \frac{16}{\nu\kappa}, \qquad Pr = \frac{\nu}{\kappa}$$

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i.e.

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$Pr \approx 0.02$ (liquid Gallium), 0.7 (air), 7 (water), $100 \leq$ (motor oil)

Boundary conditions

- Isothermal: $T(\pm 1) = \mp 1$
- No-slip: $u = v = 0$ at $y = \pm 1$
- Periodic, wavelength $\frac{2\pi}{\alpha}$ in horizontal x direction

Eliminating pressure and restricting to 2D

$$\left(\partial_t - \nu\nabla^2\right)\nabla^2 v = \partial_x\left(v\nabla^2 u - u\nabla^2 v\right) + \partial_x^2 T,$$

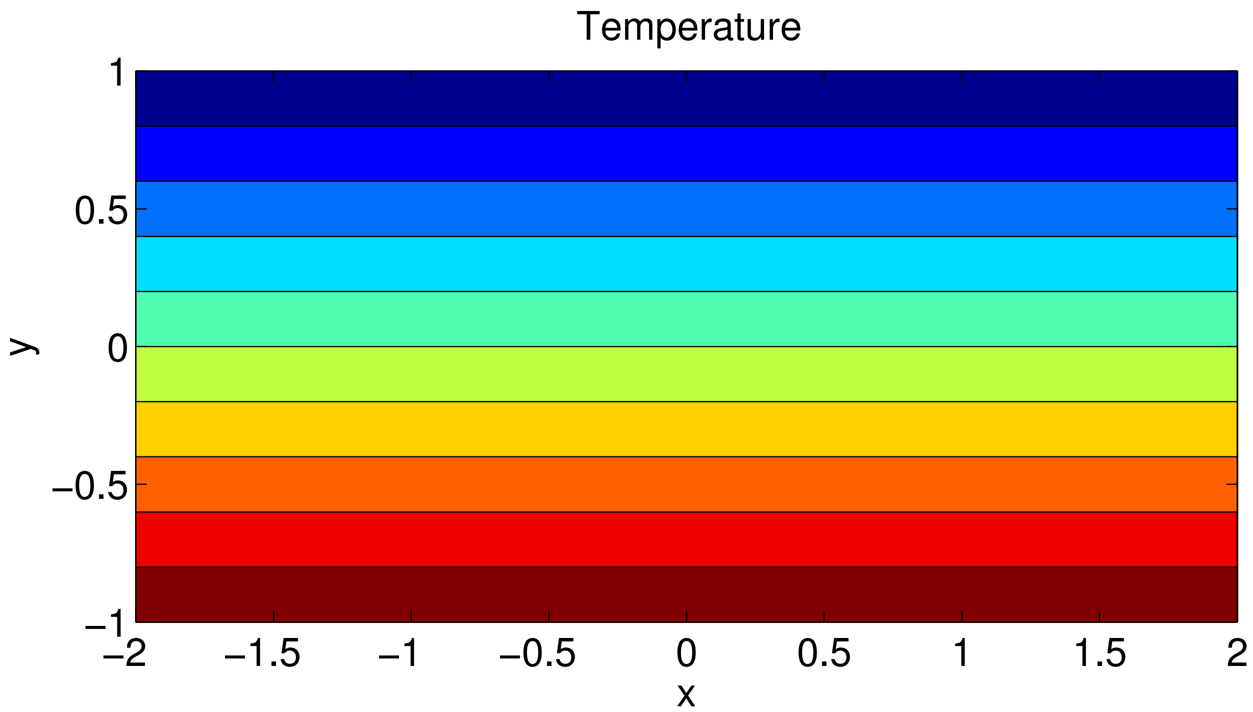
$$\left(\partial_t - \kappa\,\nabla^2\right) T = -(u\partial_x + v\partial_y)T,$$

$\partial_x u = -\partial_y v$ and eqn for mean velocity field $u_0(y,t)$:

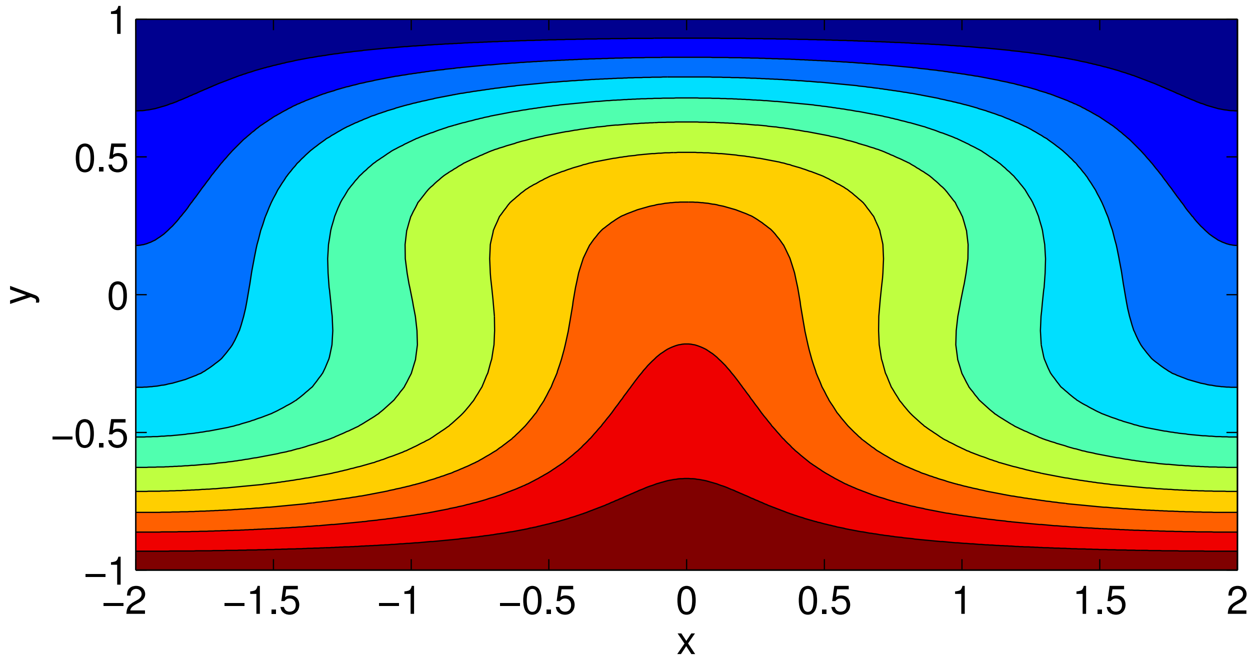
$$\left(\partial_t - \nu\,\partial_y^2\right) u_0 = -\partial_y \overline{uv}$$

BC: $v = \partial_y v = 0$, $T = \mp 1$ at $y = \pm 1$. Period $L_x = \frac{2\pi}{\alpha}$ in x

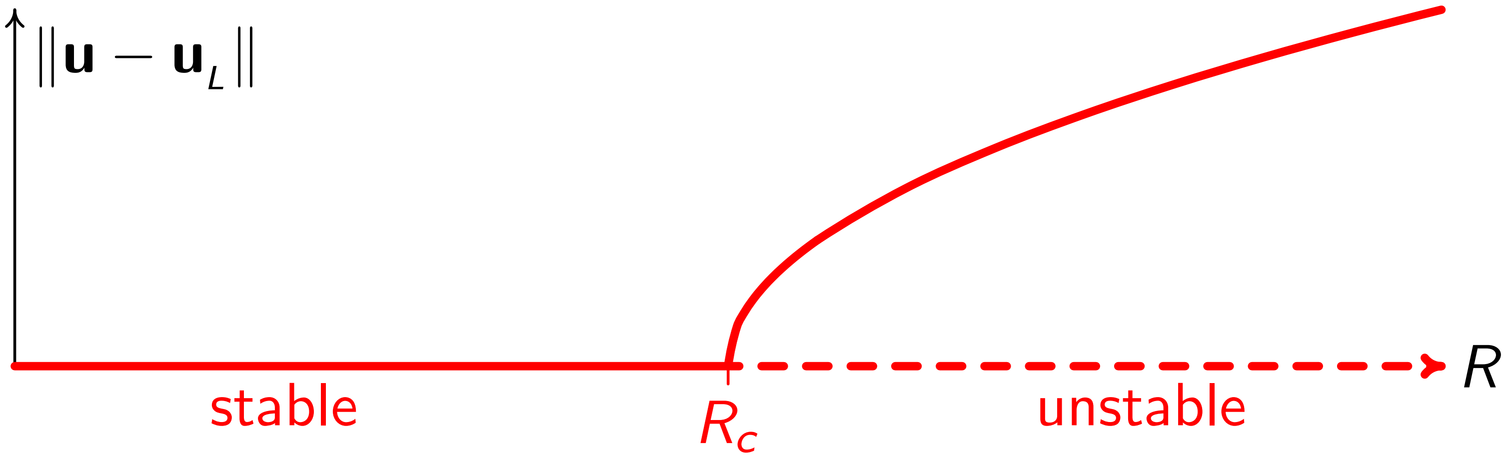
Pure conduction solution: $\boldsymbol{u} = 0,\ T = -y \qquad \forall Ra, Pr$



Bifurcation to convection for $Ra > 1708$, $\alpha_c \approx \pi/2$, $\forall Pr$



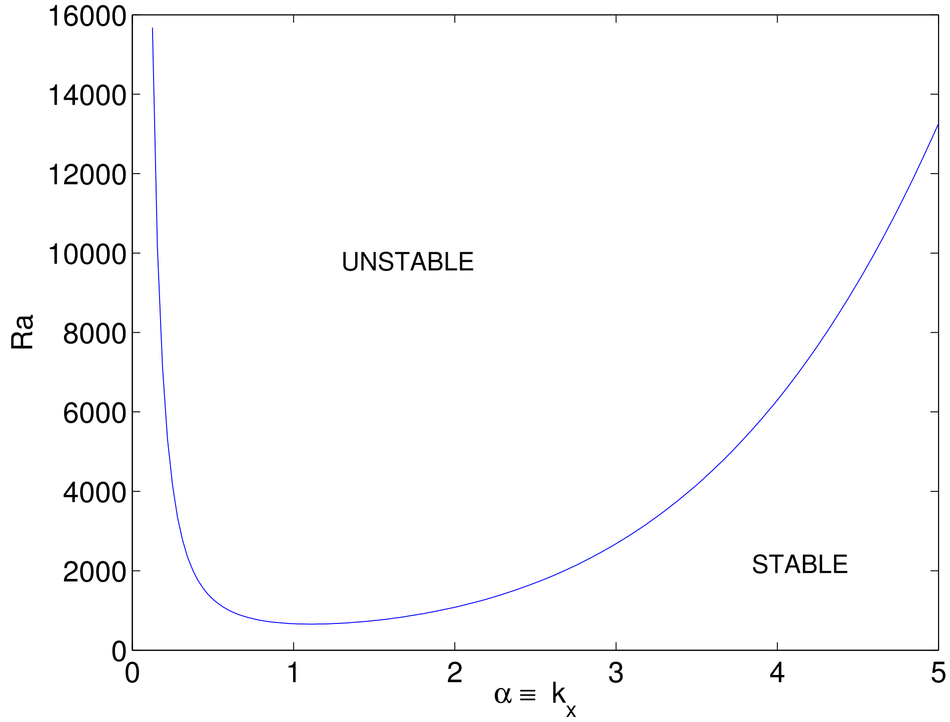
supercritical bifurcation from $Ra = 1708$ (no-slip)



Stuart-Landau equation:

$$\frac{dA}{dt} \simeq (R - R_c)A - \lambda_2 |A|^2 A + \cdots$$

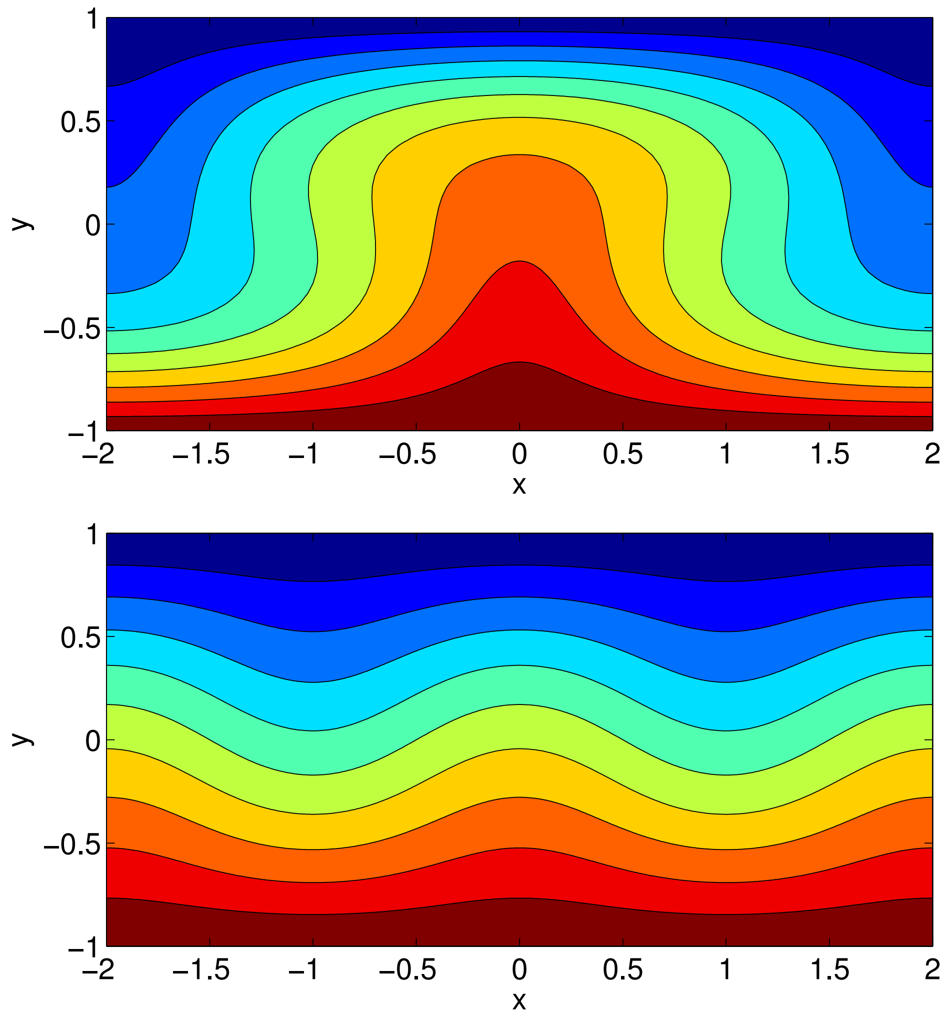
stability curve about: $\boldsymbol{u} = 0$, $T = -y$



$$Ra_c = 16\,\frac{(\alpha^2 + \pi^2/4)^3}{\alpha^2} \quad \text{free-slip}$$

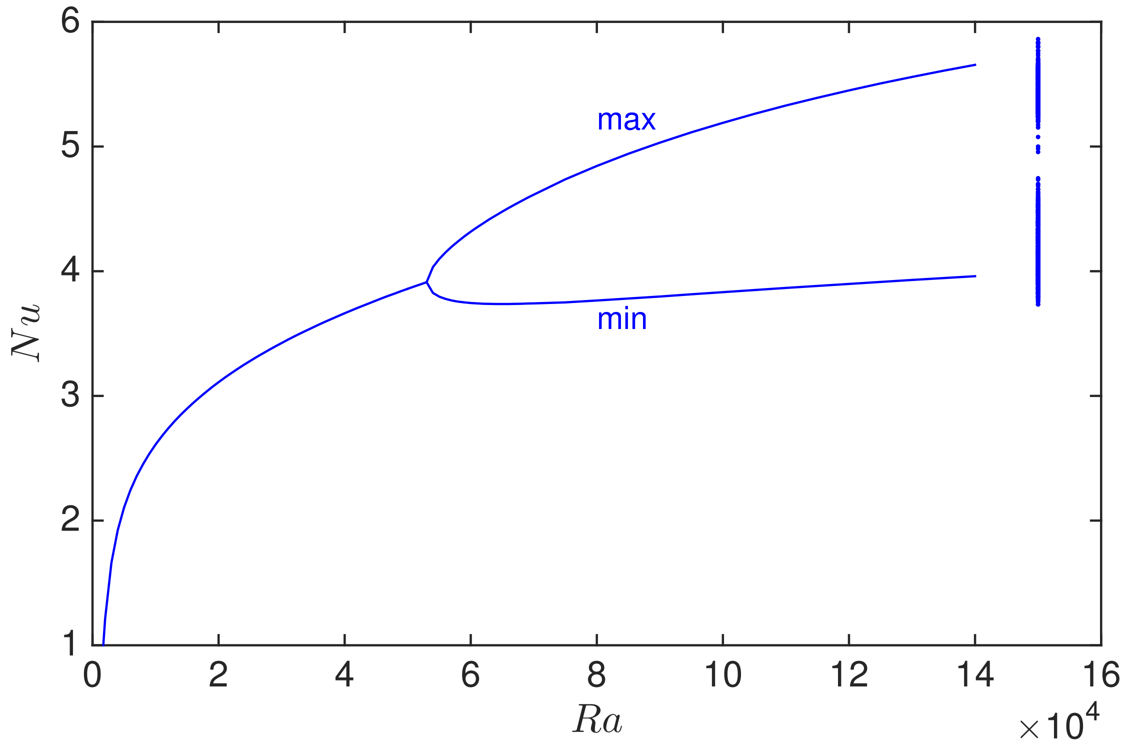
Qualitatively same for no-slip: $\alpha_c \approx 1.558$, $Ra_c \approx 1708$.
Independent of Pr (only $\nu\kappa$ matters for linear stability).

Multiple convective equilibria for $L/H = 2$, $Ra = 4\,000$



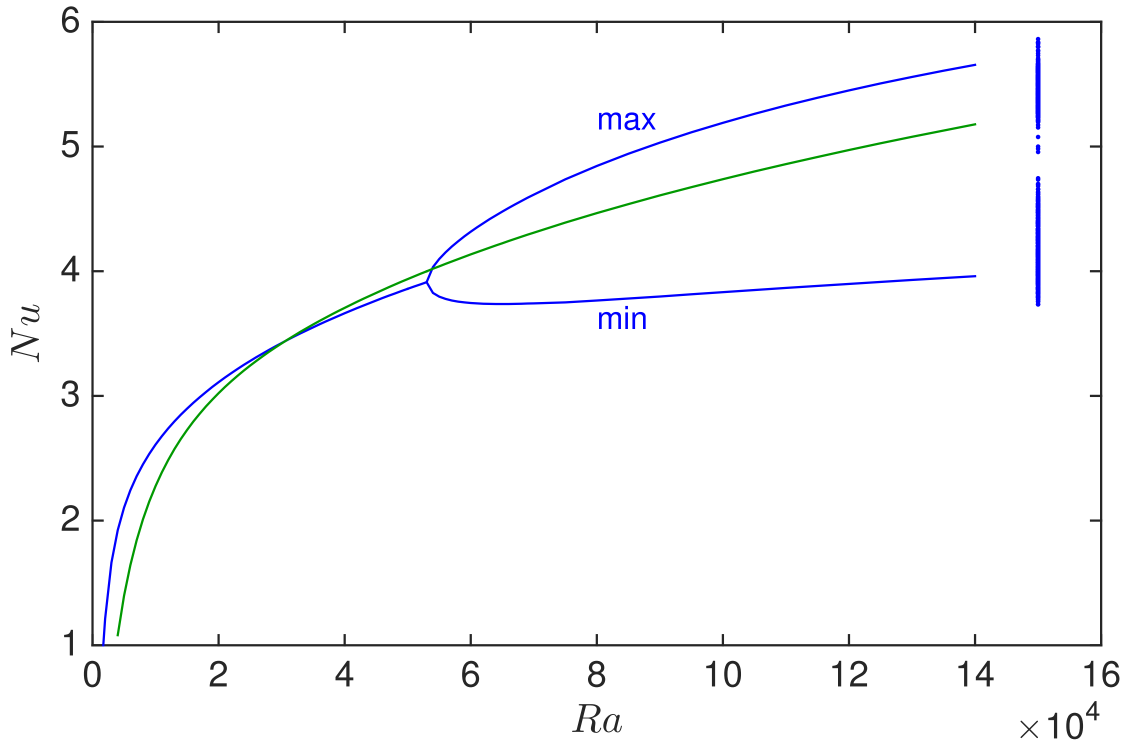
Top: primary mode $\alpha = \pi/2 \approx \alpha_c$ Bottom: $\alpha = \pi$ mode

Bifurcations of Primary Branch



Primary mode $\alpha \approx \pi/2$ (blue), Hopf bifurcation near *Ra* $\approx 53\,000$

Bifurcation of Primary 2nd harmonic



Primary mode $\alpha \approx \pi/2$ (blue), Hopf bifurcation near *Ra* $\approx 53\,000$

Second mode $\alpha = \pi$ (green)

Heat transport by marginal stability (Malkus 1954)

- Interior at $T = 0$: marginal inviscid stability

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$$Ra^{(\delta)} = \frac{g\alpha(\Delta T/2)\delta^3}{\nu\kappa} \approx \frac{1708}{16}$$

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-

$$\Rightarrow \boxed{Nu \sim \frac{\kappa(\Delta T/2)/\delta}{\kappa(\Delta T)/H} \sim 0.084\, Ra^{1/3}}$$

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-

$$\Rightarrow \boxed{Nu \sim \frac{\kappa(\Delta T/2)/\delta}{\kappa(\Delta T)/H} \sim 0.084\, Ra^{1/3}}$$

- flux independent of H (Pr too).

- What do unstable coherent states have to tell us?

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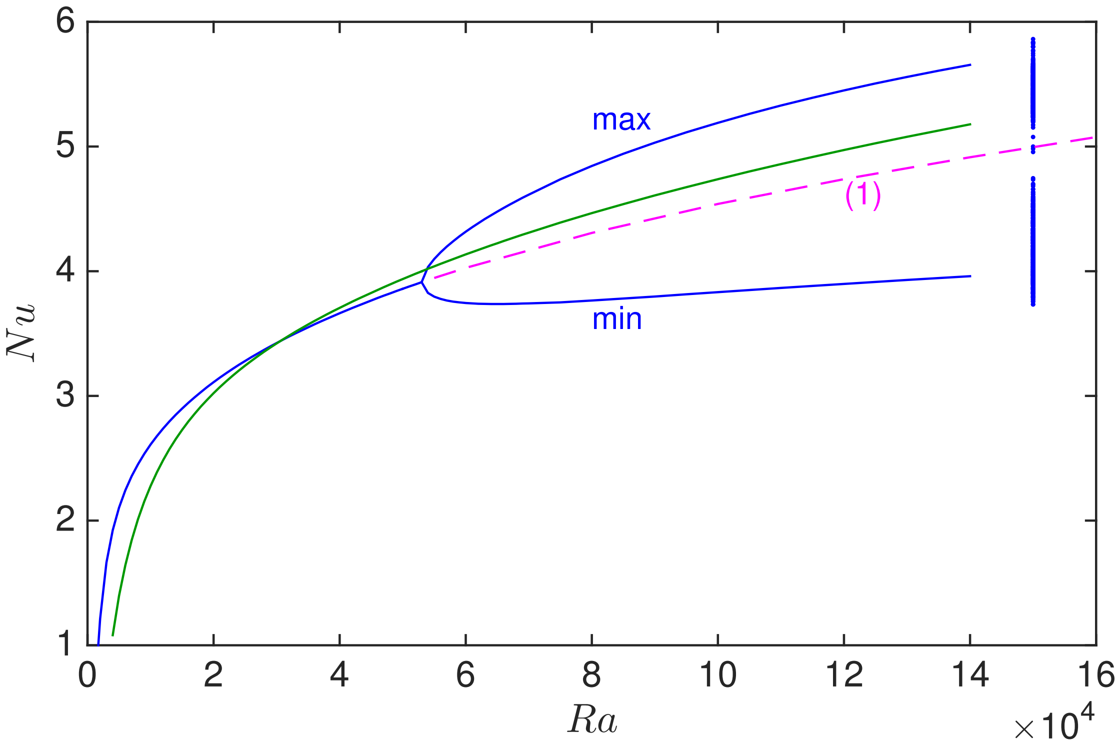
- Which unstable coherent states should we consider?

Continuation of unstable Primary (1)



Primary mode $\alpha \approx \pi/2$ (blue), Hopf bifurcation near *Ra* $\approx$ 53 000

Second mode $\alpha = \pi$ (green)

Compute

- *2D steady states*, typically *unstable*
- Impose mirror symmetry

$$[u, v, T](x, y, t) = [-u, v, T](-x, y, t)$$

$$\Rightarrow u_0(y, t) = 0 \quad \text{no mean shear}$$

- Shift-reflect symmetry (Newton only)

$$[u, v, T](x, y, t) = [u, -v, -T](x + \frac{L_x}{2}, -y, t)$$

Numerical, spectral expansion

Chebyshev polynomials $T_m(y)$ *integration* in wall-normal y direction, Fourier in x (periodic)

$$\partial_y^4 v(x,y,t) = \sum_{l=-L_T}^{L_T} \sum_{m=0}^{N_C} a_{lm}(t) T_m(y) e^{il\alpha x},$$

$$\partial_y^2 T(x,y,t) = \sum_{l=-L_T}^{L_T} \sum_{m=0}^{N_C} b_{lm}(t) T_m(y) e^{il\alpha x},$$

Zebib 1984 for OS, Greengard 1989 for Heat, Jeffreys 1928 for RBC! ... *Four* separate codes, differ in treatment of 4th order v equation, different time-integration schemes, symmetries, Newton, etc.

Numerics

- Cheb-Tau (or collocation) doesn't work at all for time-marching (huge spurious eigs > 0)
- Cheb-Galerkin $p = 1$ or 2 fine

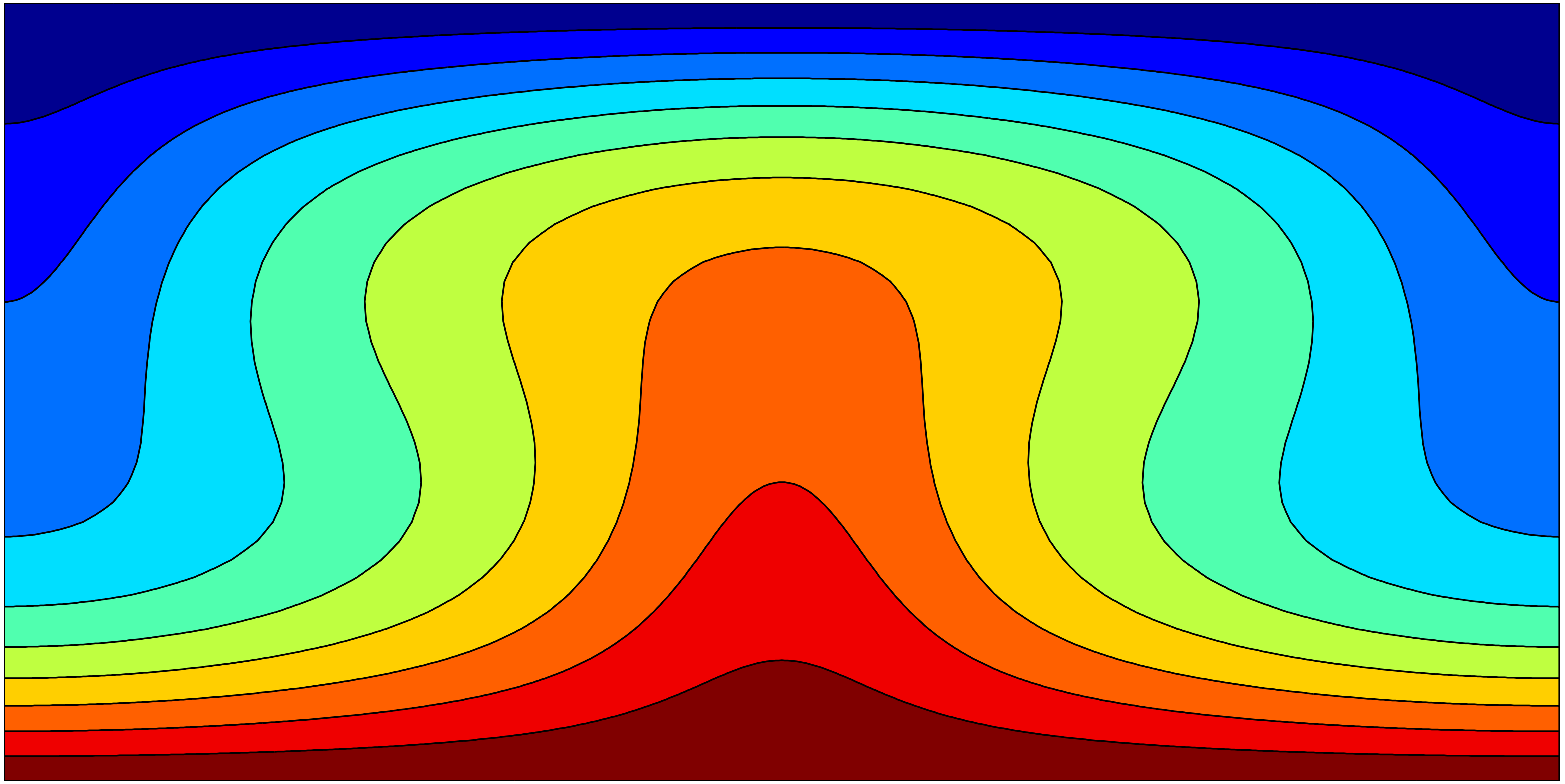
$$\int_{-1}^{1} f(y)\,(1-y^2)^p \frac{T_m(y)}{\sqrt{1-y^2}}\,dy$$

- (Chebyshev $\to$ Gegenbauer/Ultraspherical)
- Proven!

  M. Charalambides and FW, SIAM J. Numer. Analysis 2008

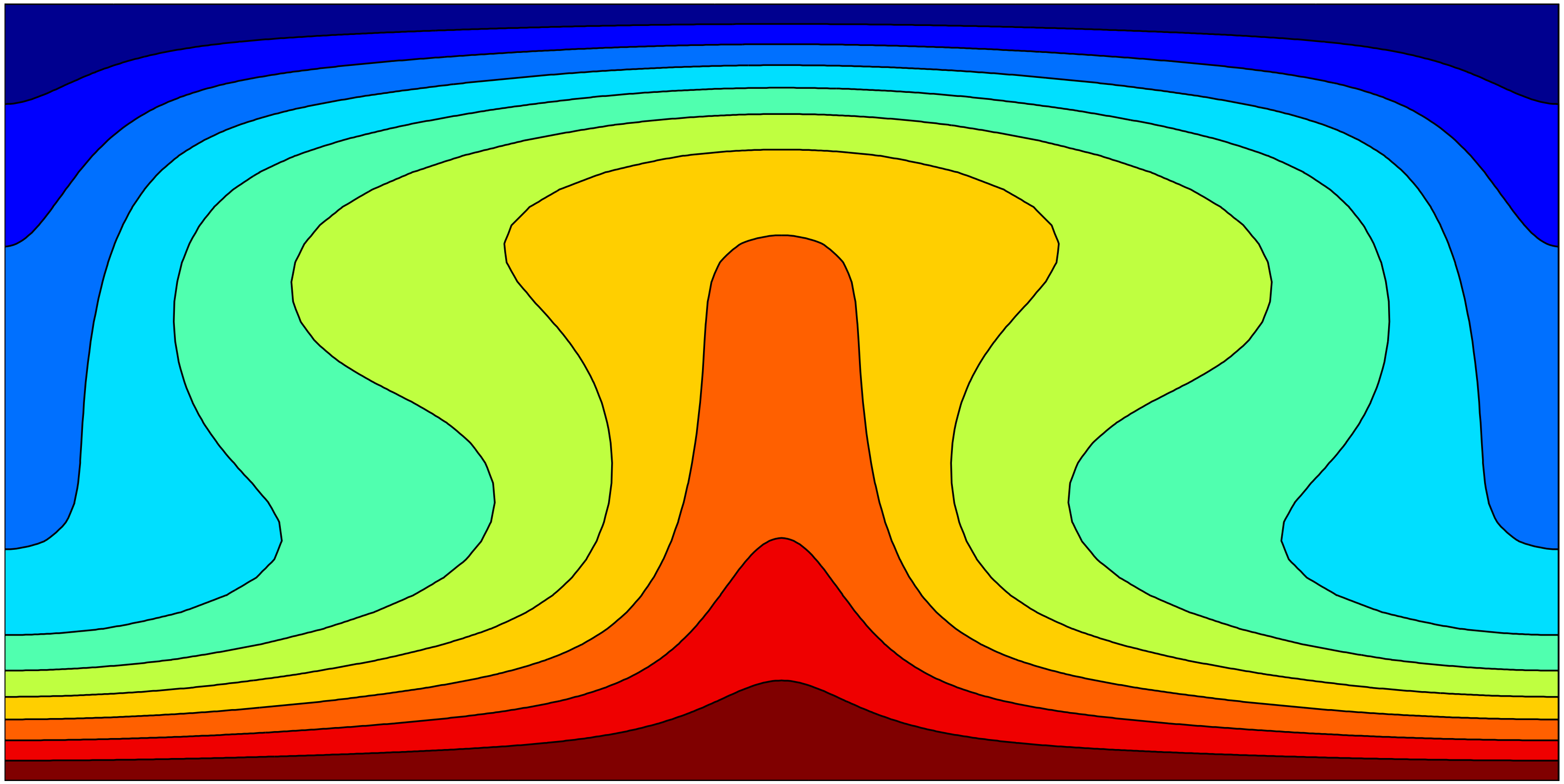
(1) Primary branch, $L/H \simeq 2$, unstable for $Ra > 53\,000$

$Ra = 5\,000$



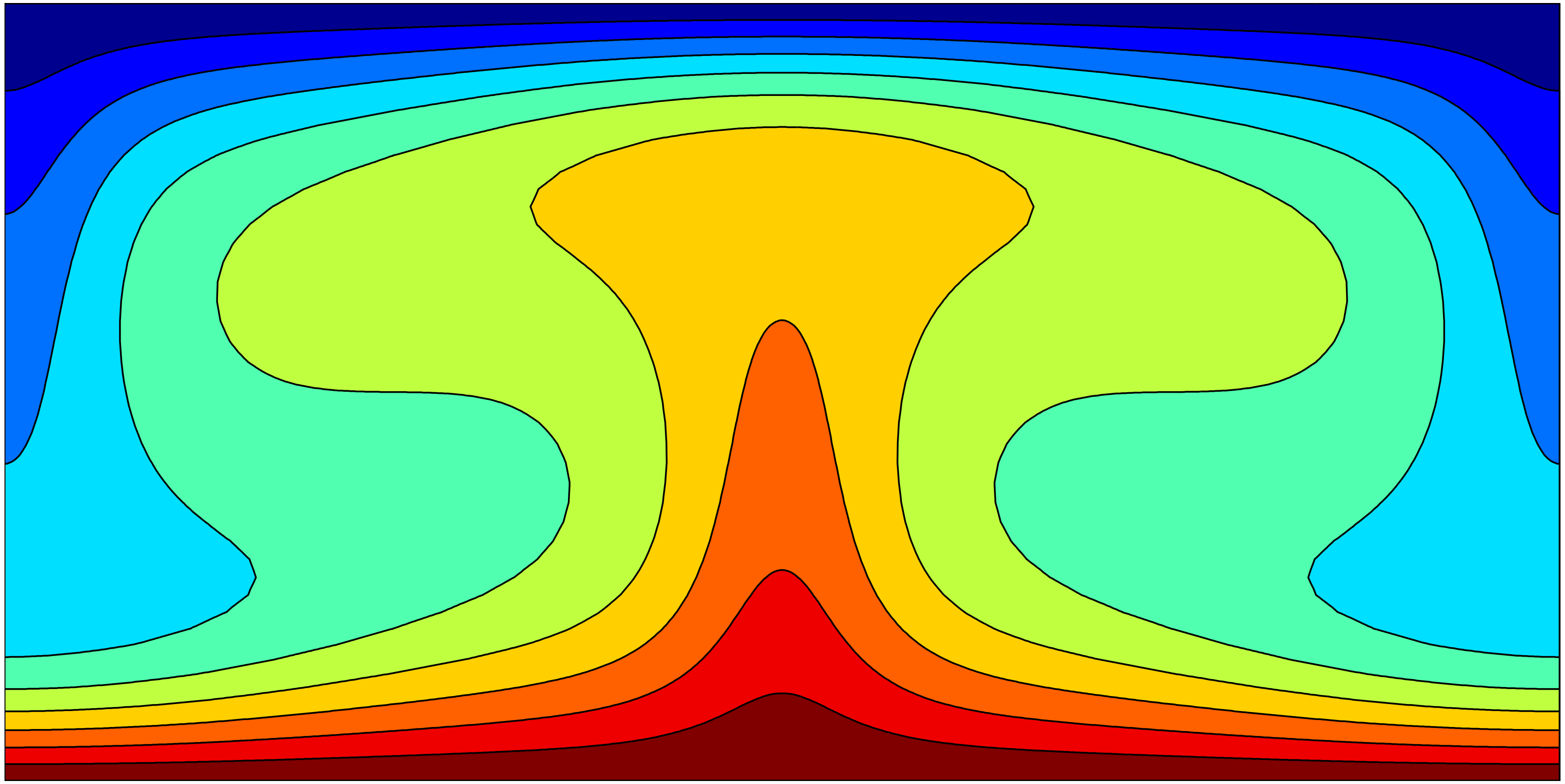
(1) Primary branch, $L/H \simeq 2$, unstable for $Ra > 53\,000$

$Ra = 10\,000$



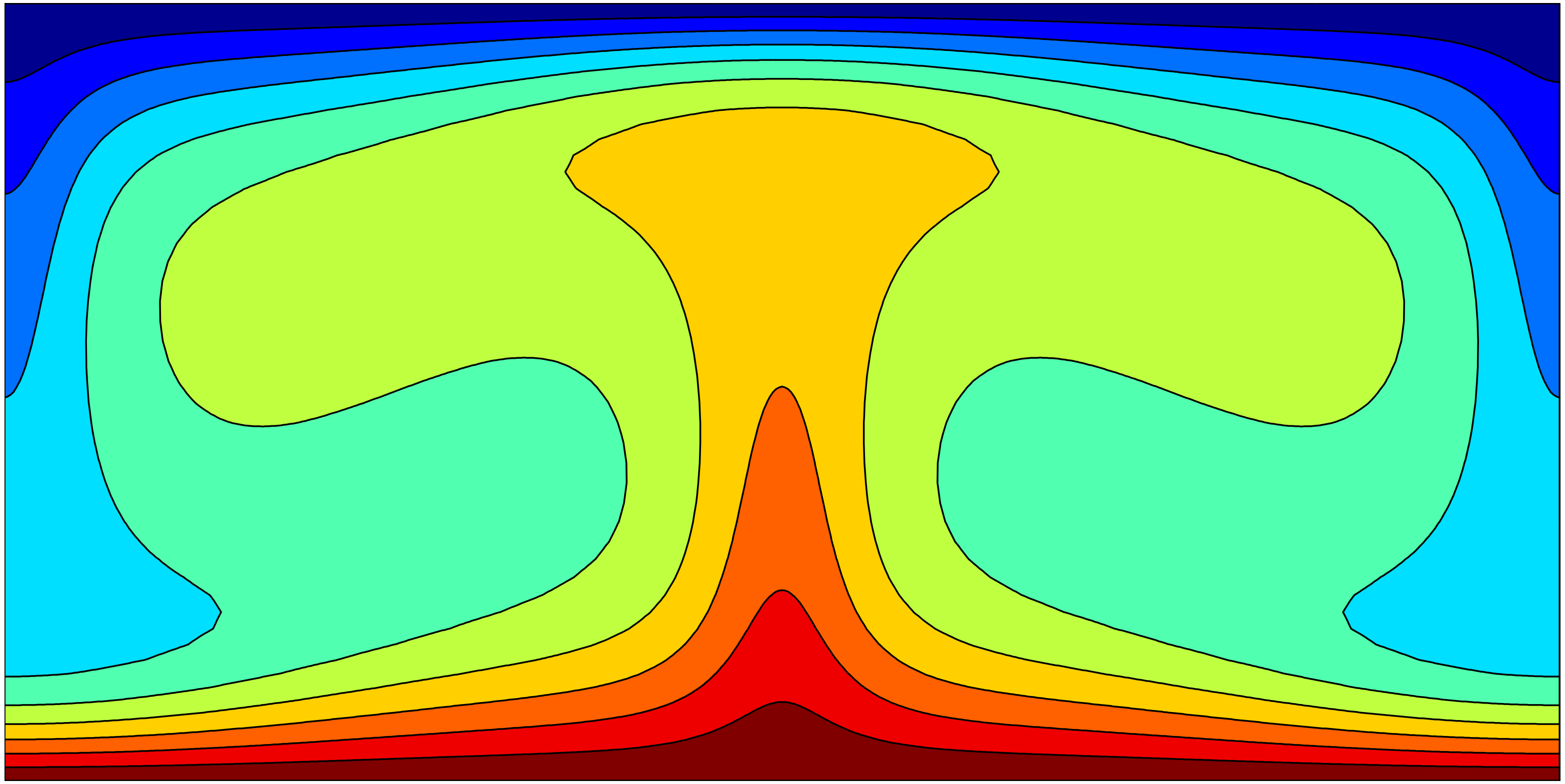
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 20\,000$



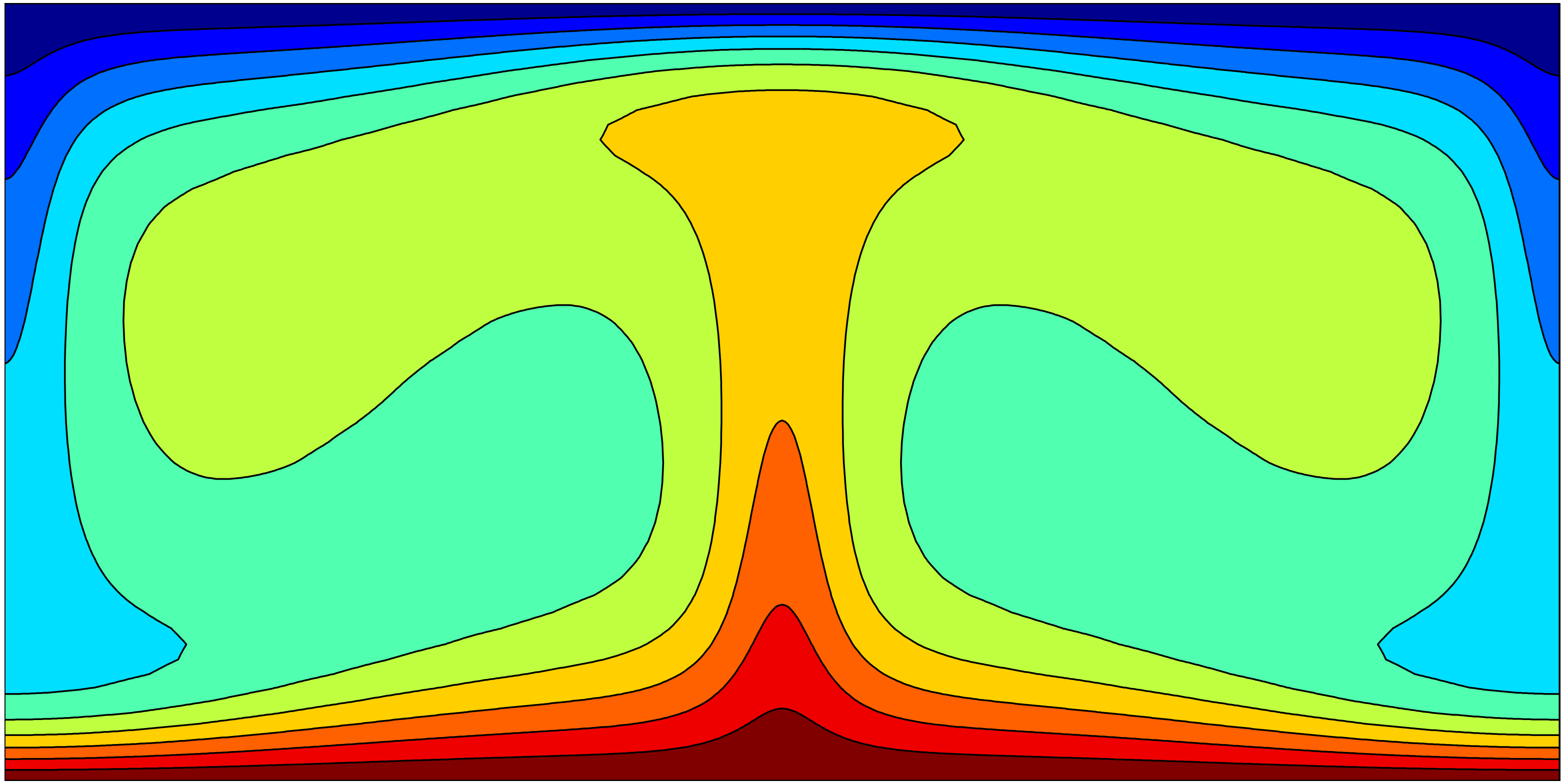
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 40\,000$



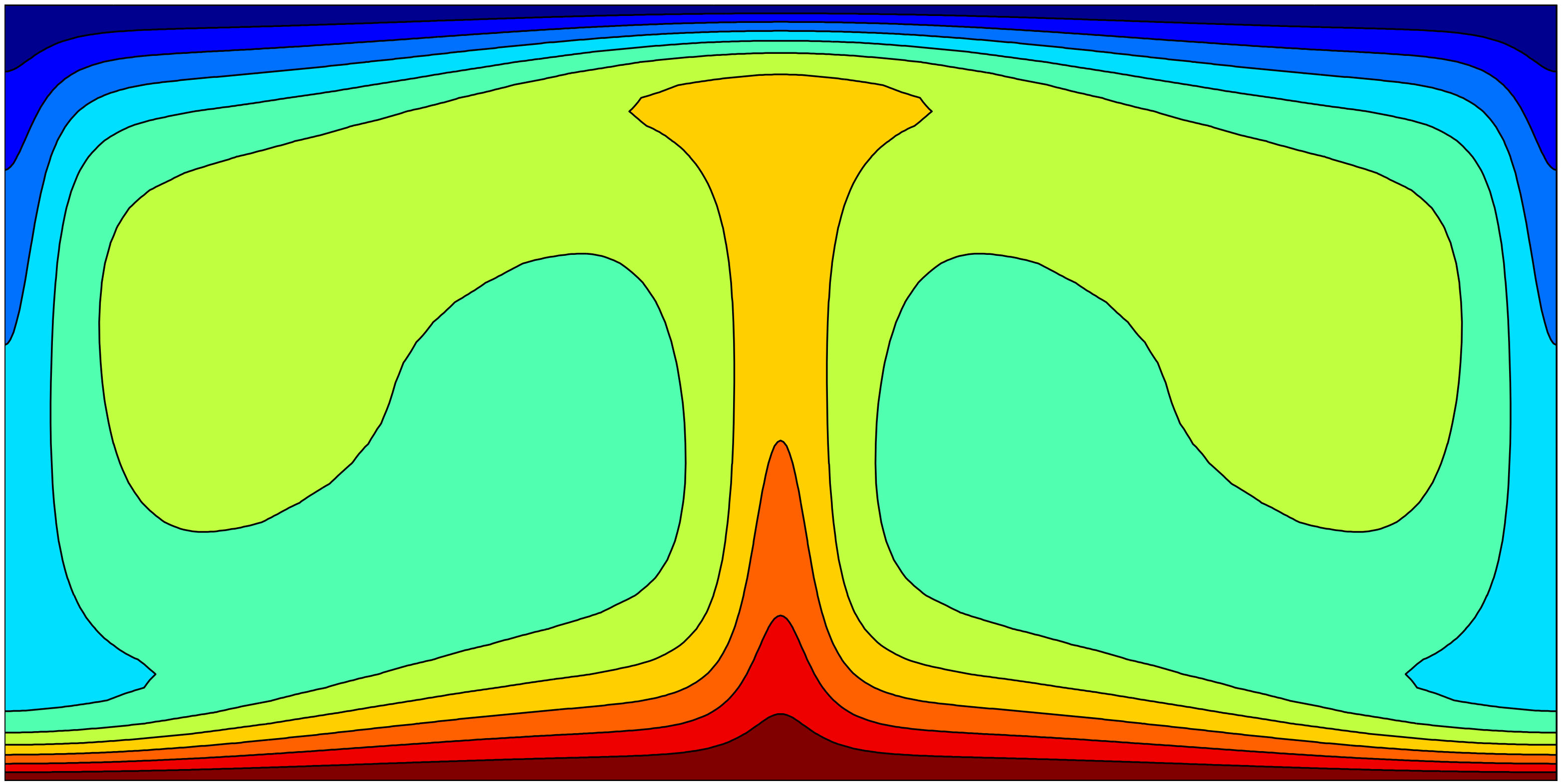
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 80\,000$



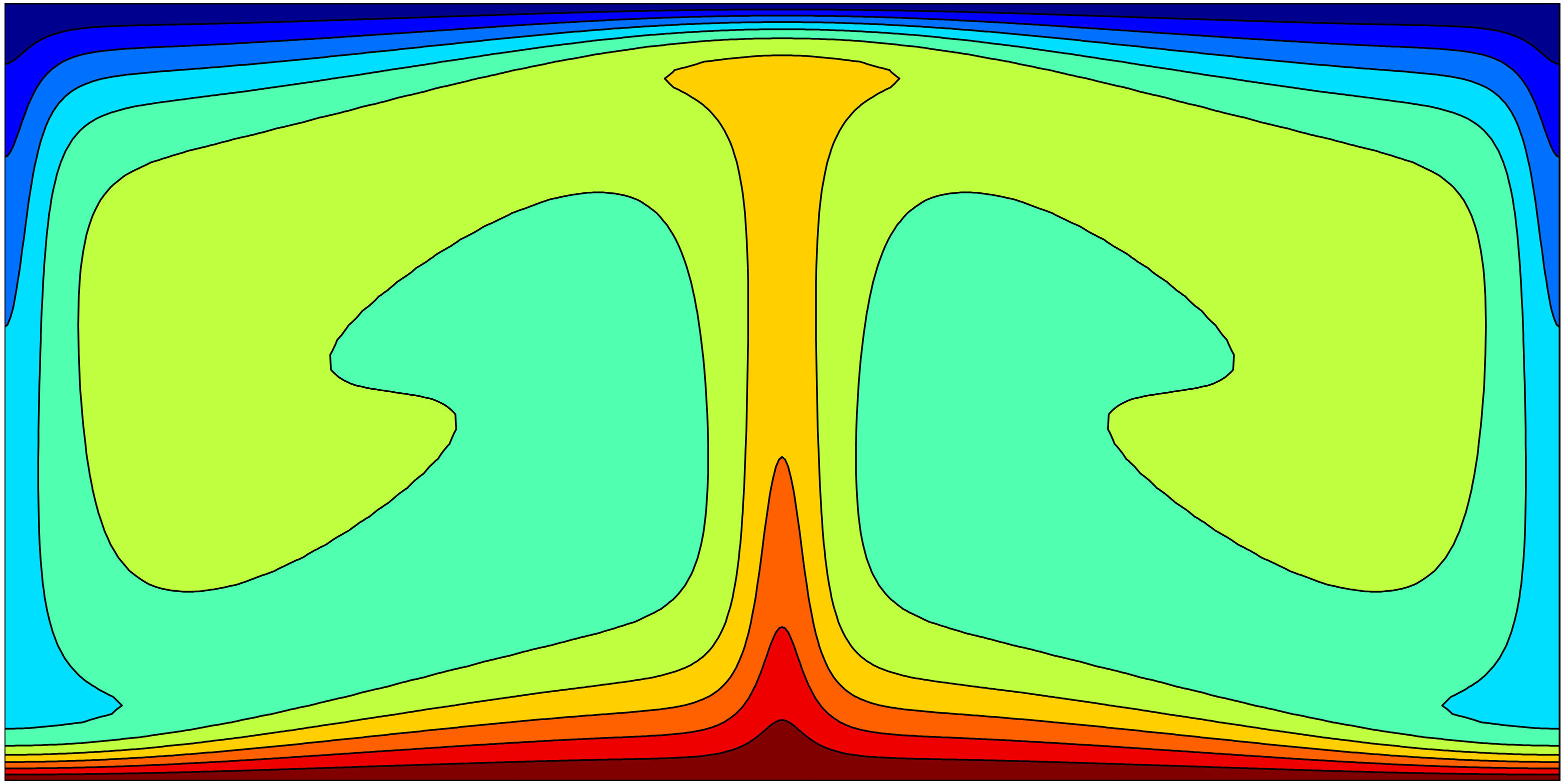
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 160\,000$



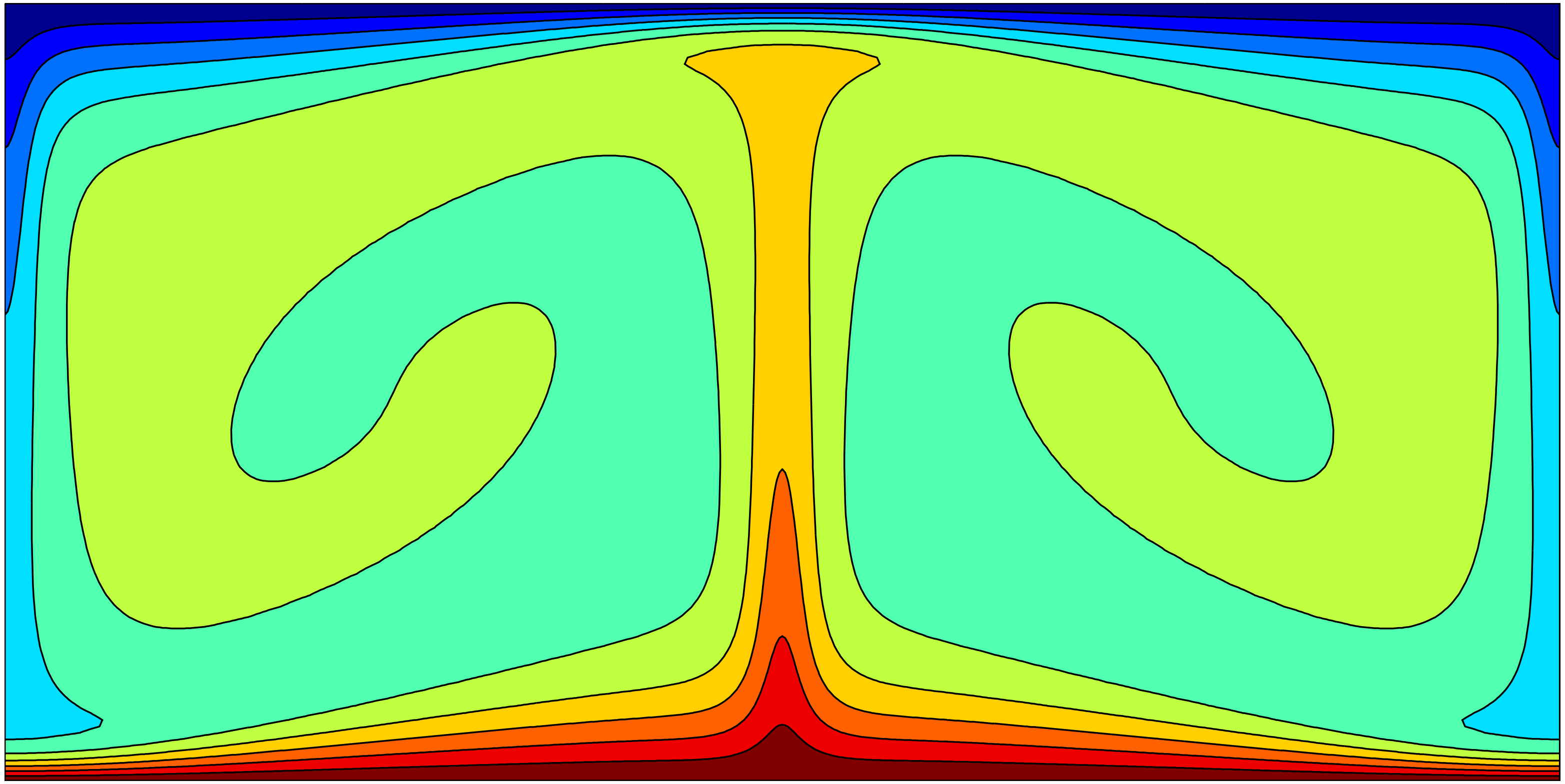
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 400\,000$



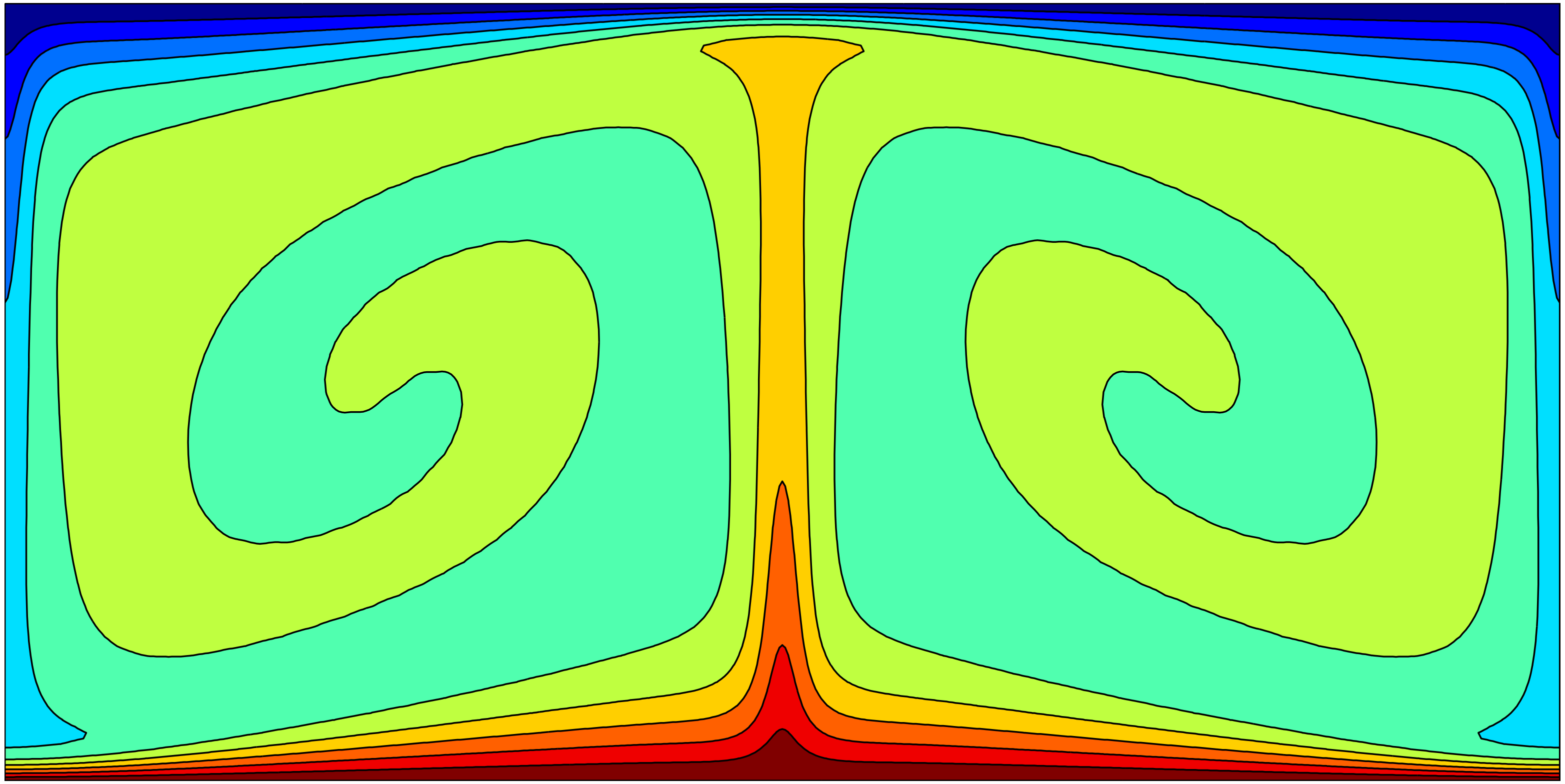
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 800\,000$



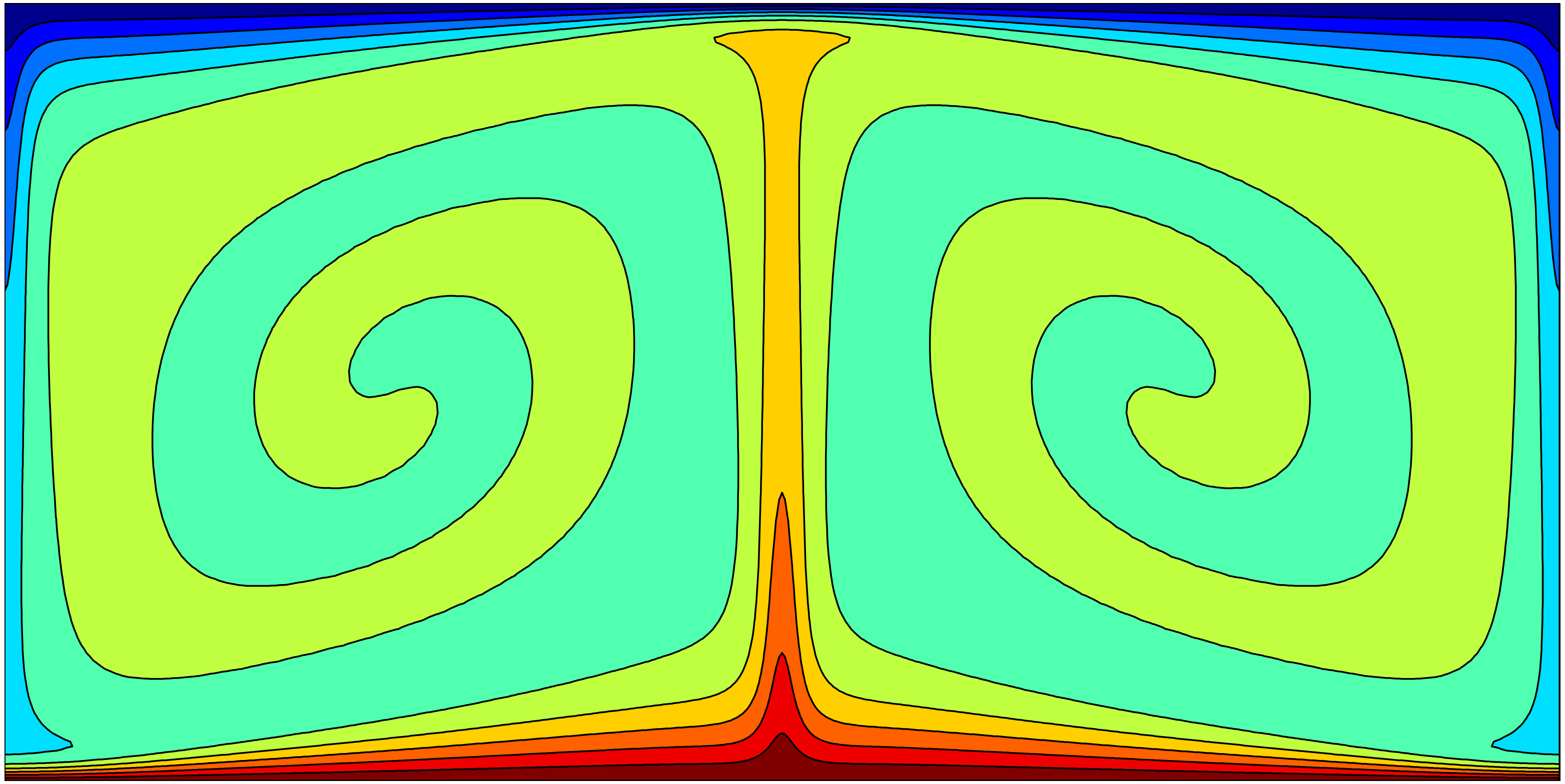
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 1\,600\,000$



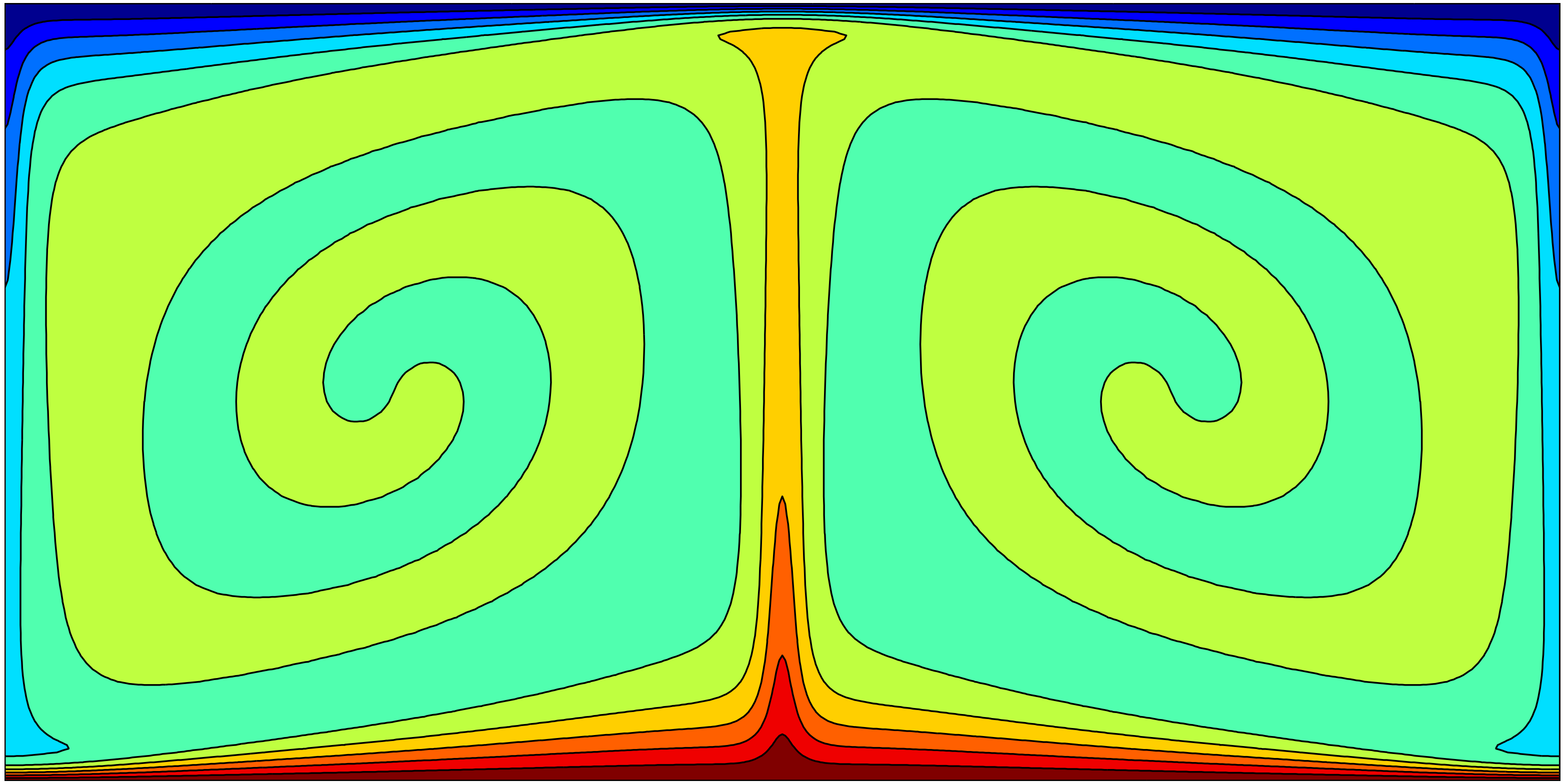
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 3\,200\,000$



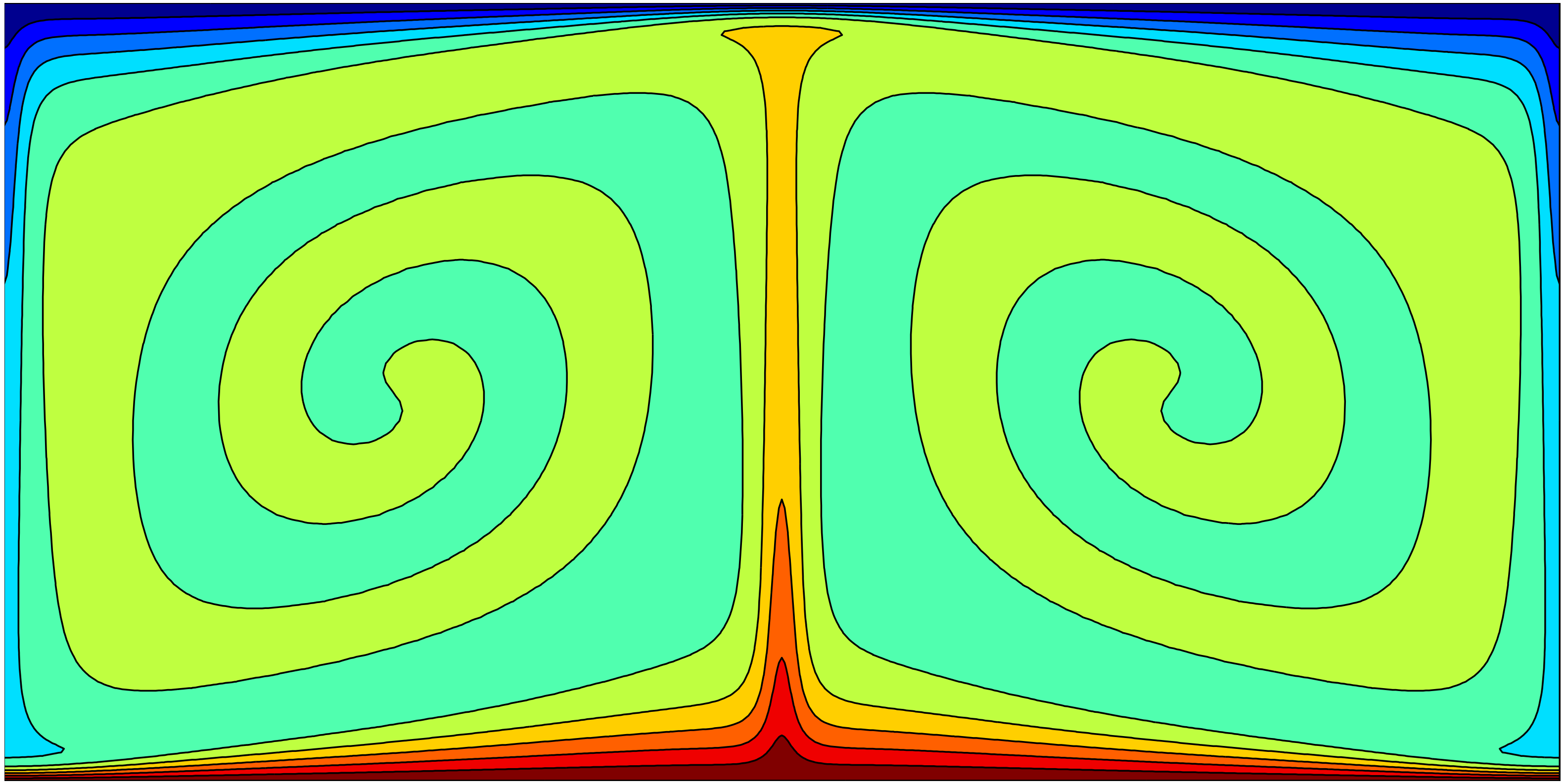
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 4\,000\,000$



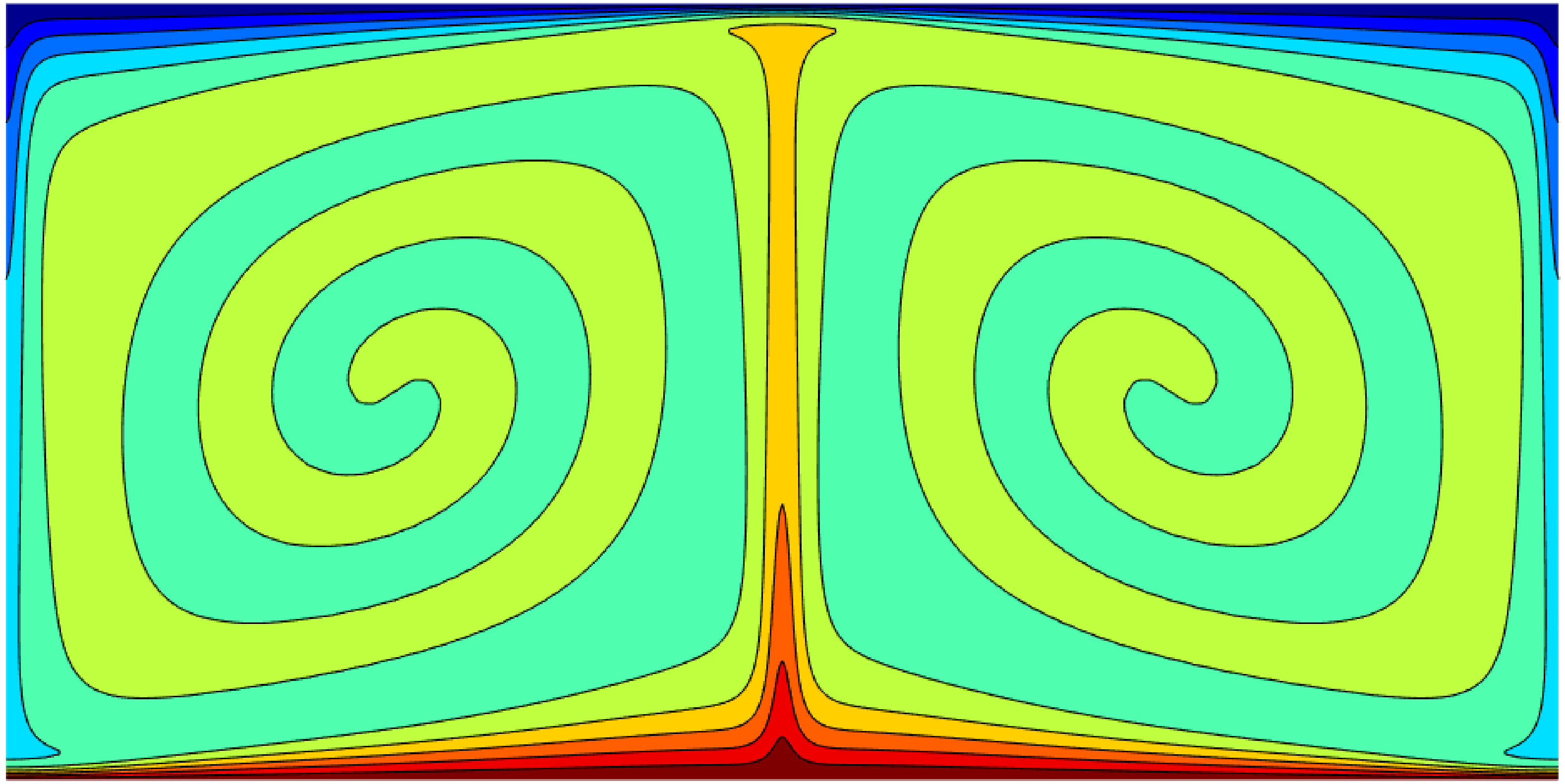
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 5\,000\,000$



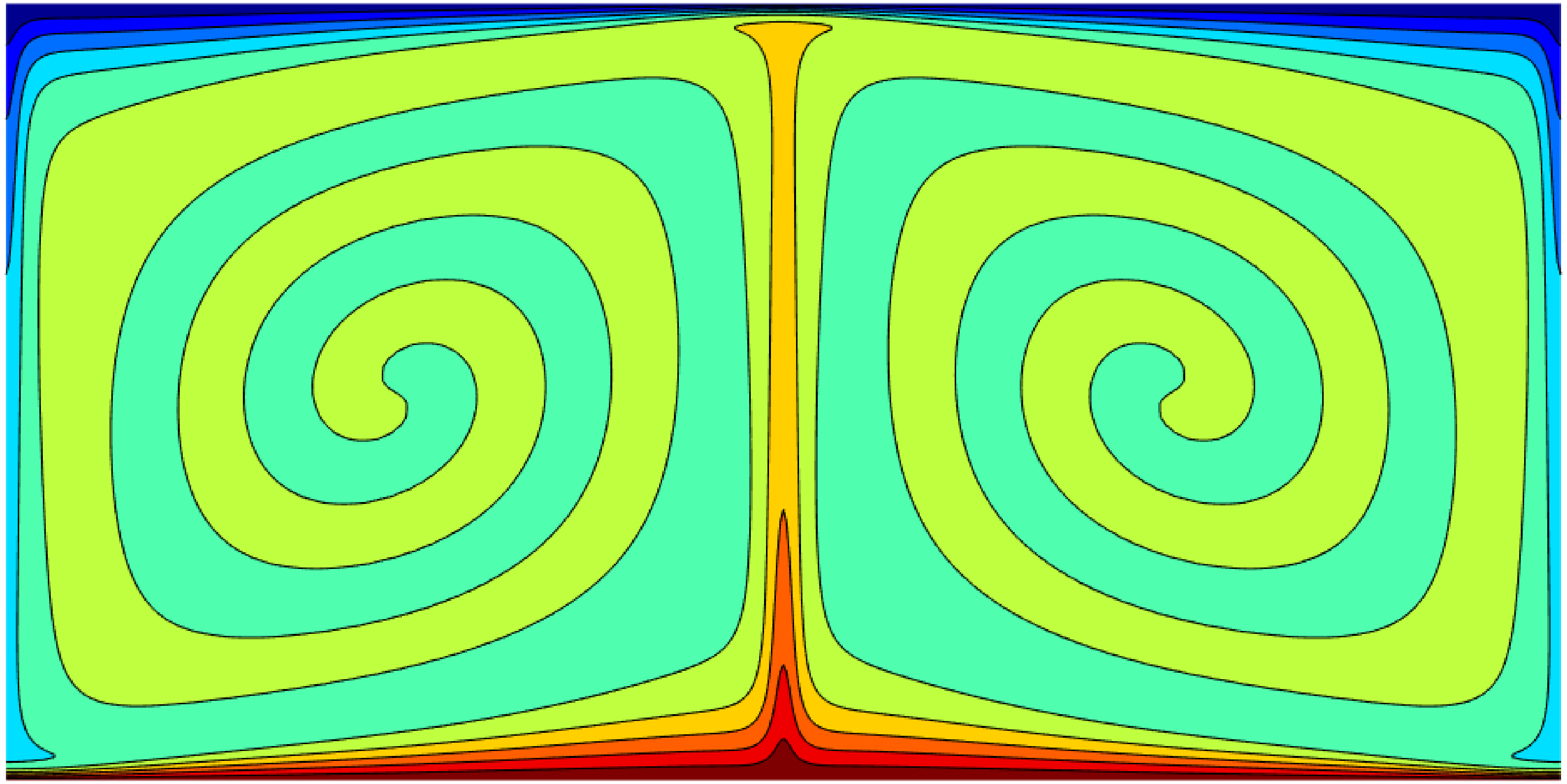
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 7\,000\,000$



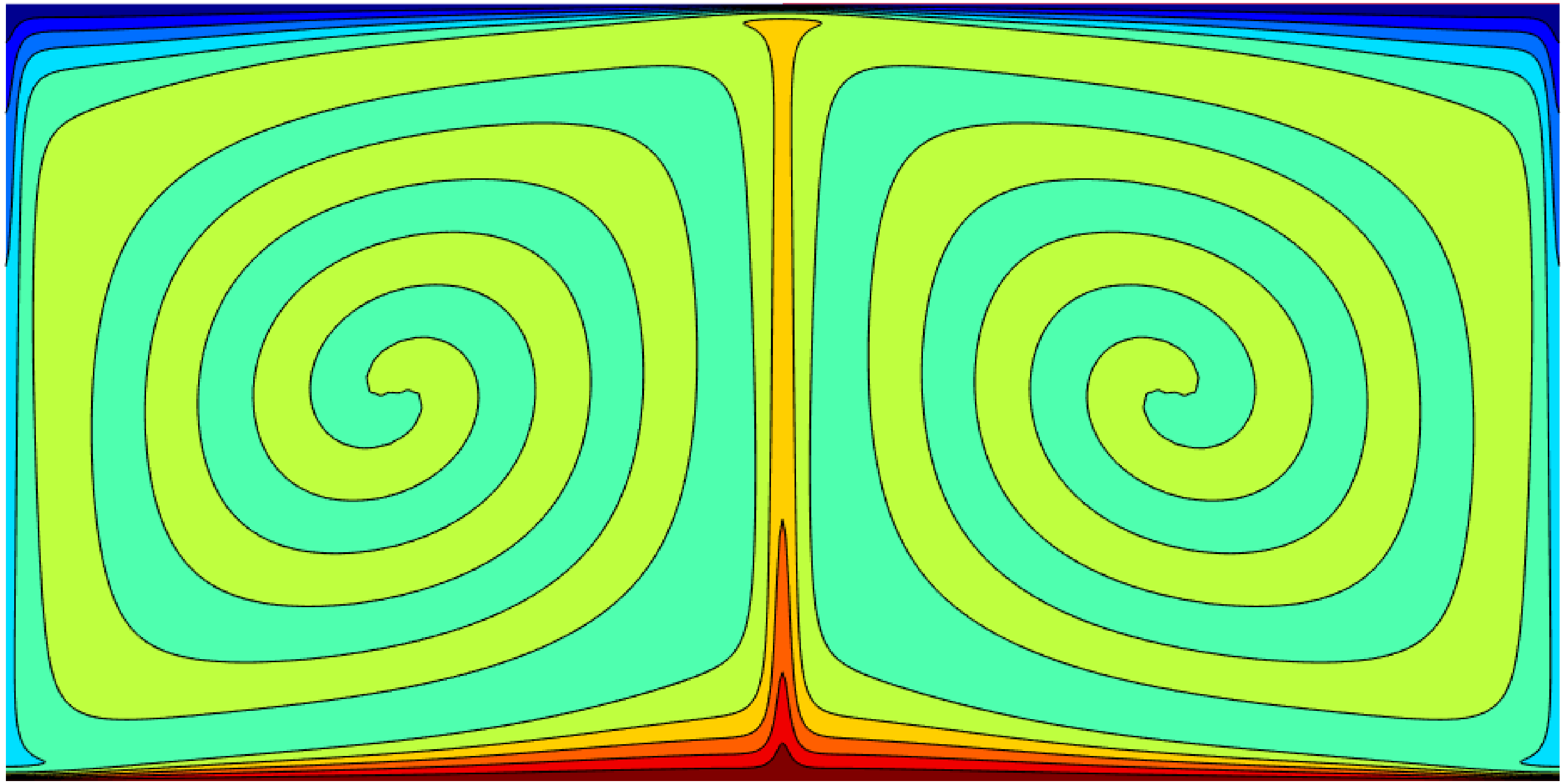
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 10\,000\,000$



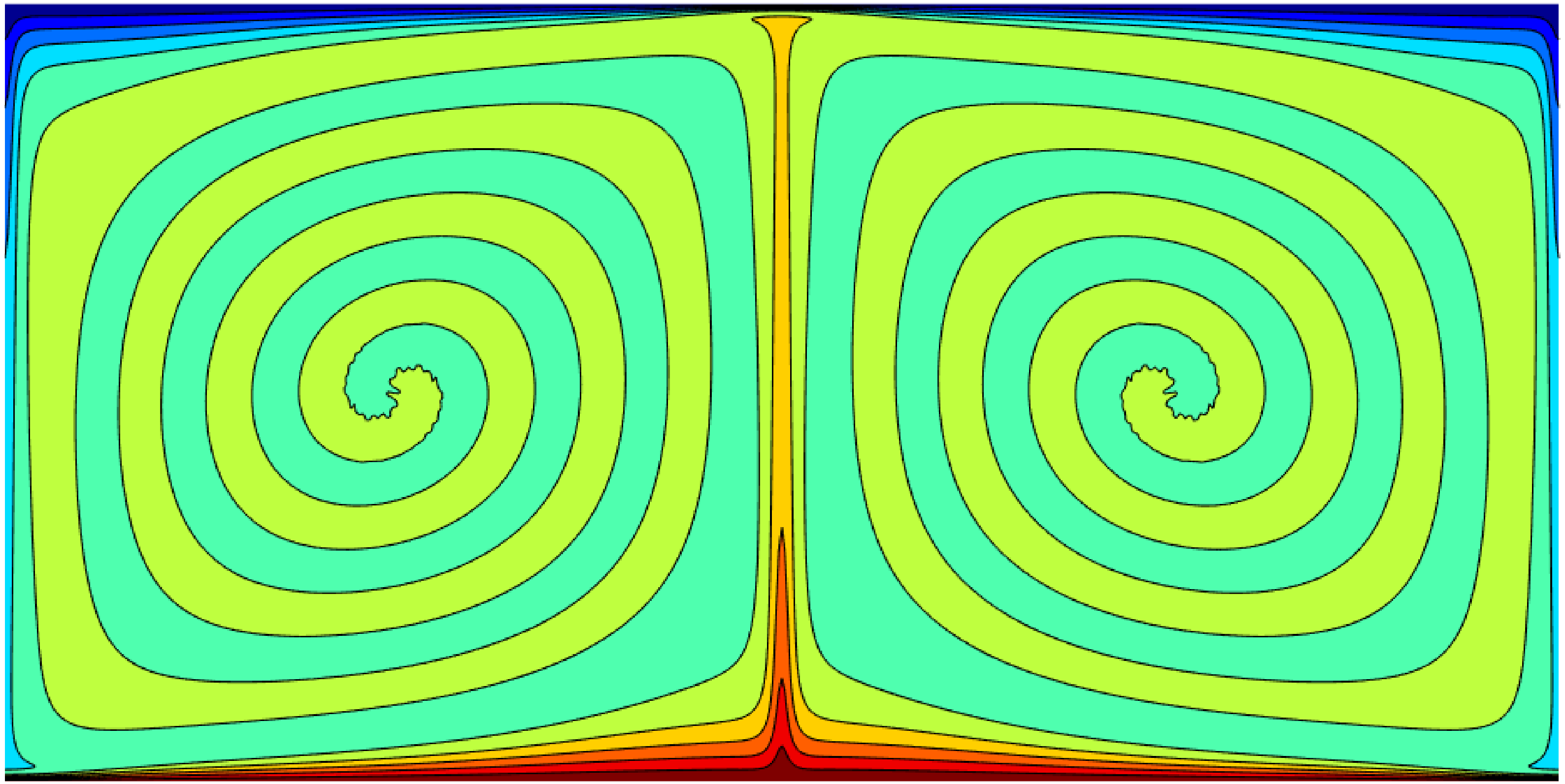
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 20\,000\,000$



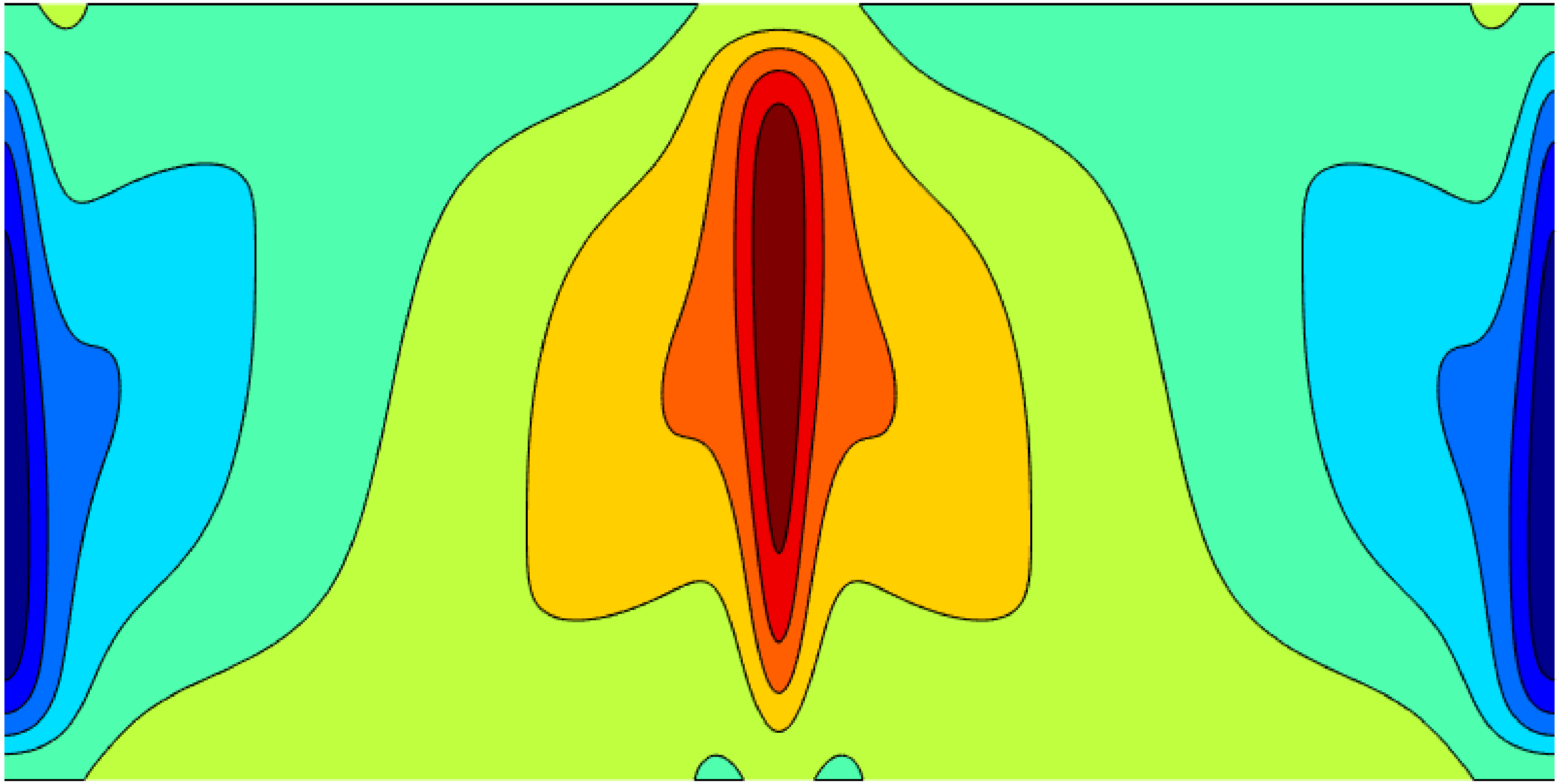
(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 40\,000\,000$



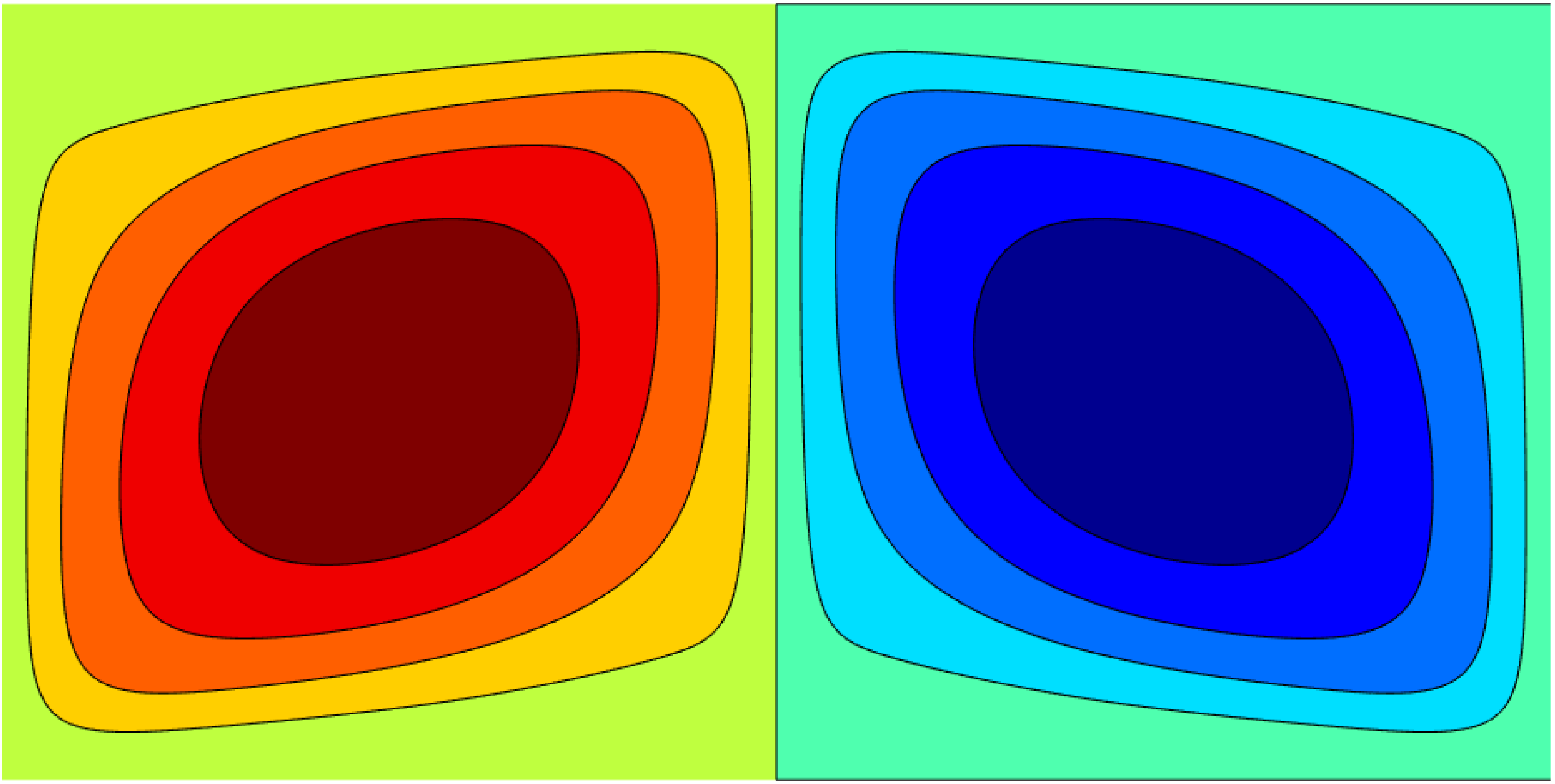
(1) Primary branch, $L/H = 2$, Velocity v

$Ra = 10\,000\,000$



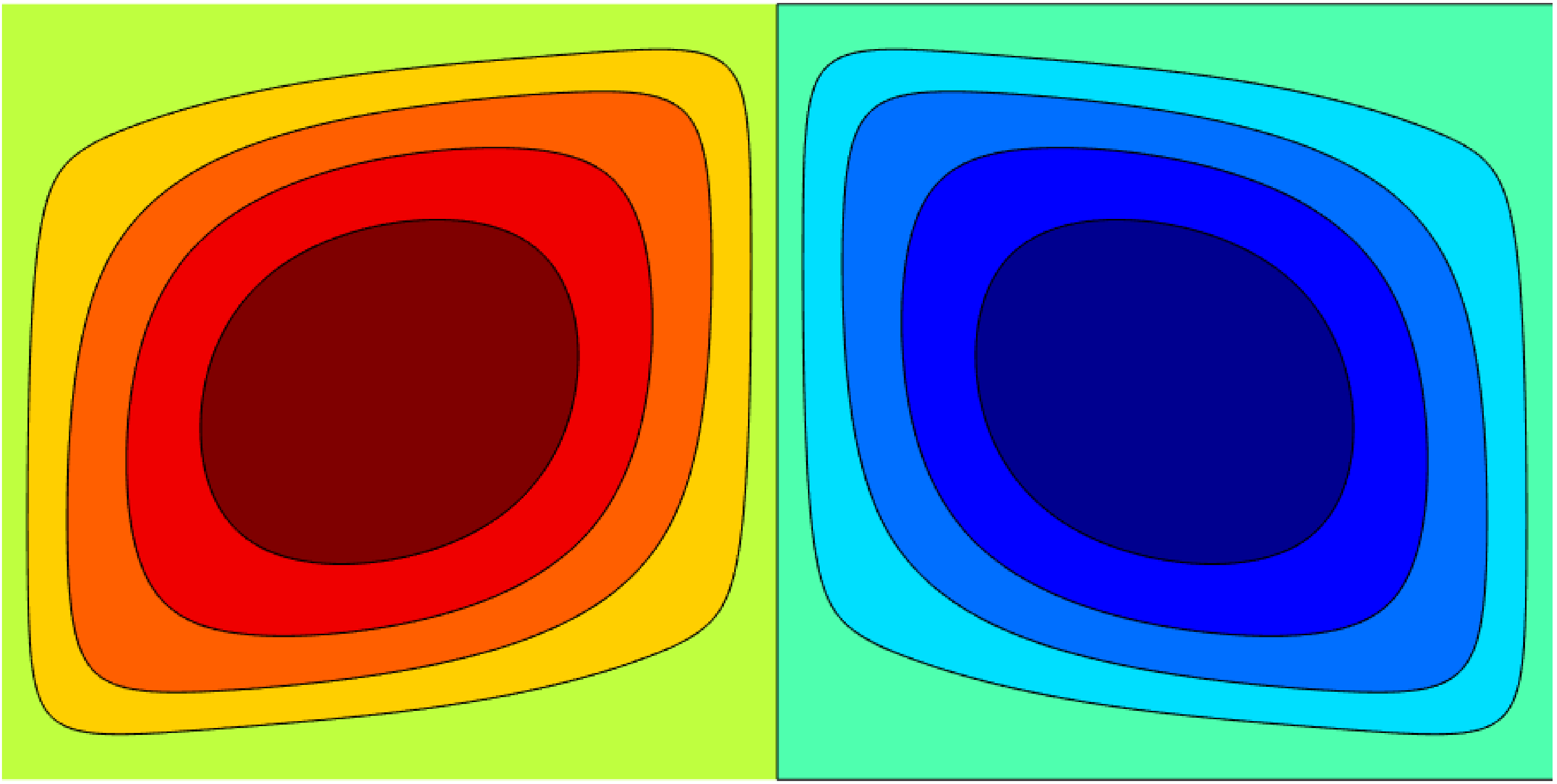
(1) Primary branch, $L/H = 2$, Streamfunction Ψ

$Ra = 5\,000\,000$



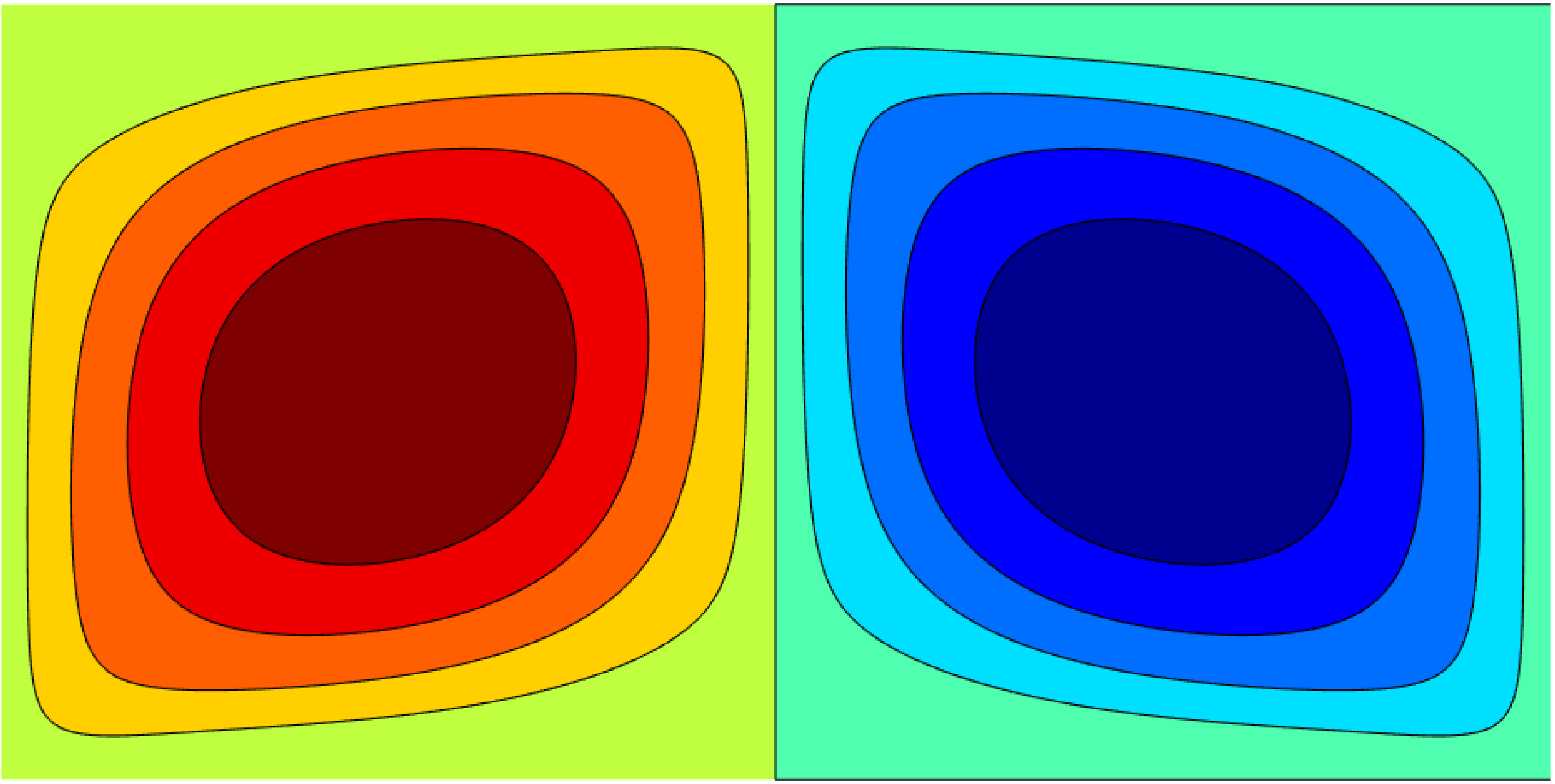
(1) Primary branch, $L/H = 2$, Streamfunction Ψ

$Ra = 10\,000\,000$



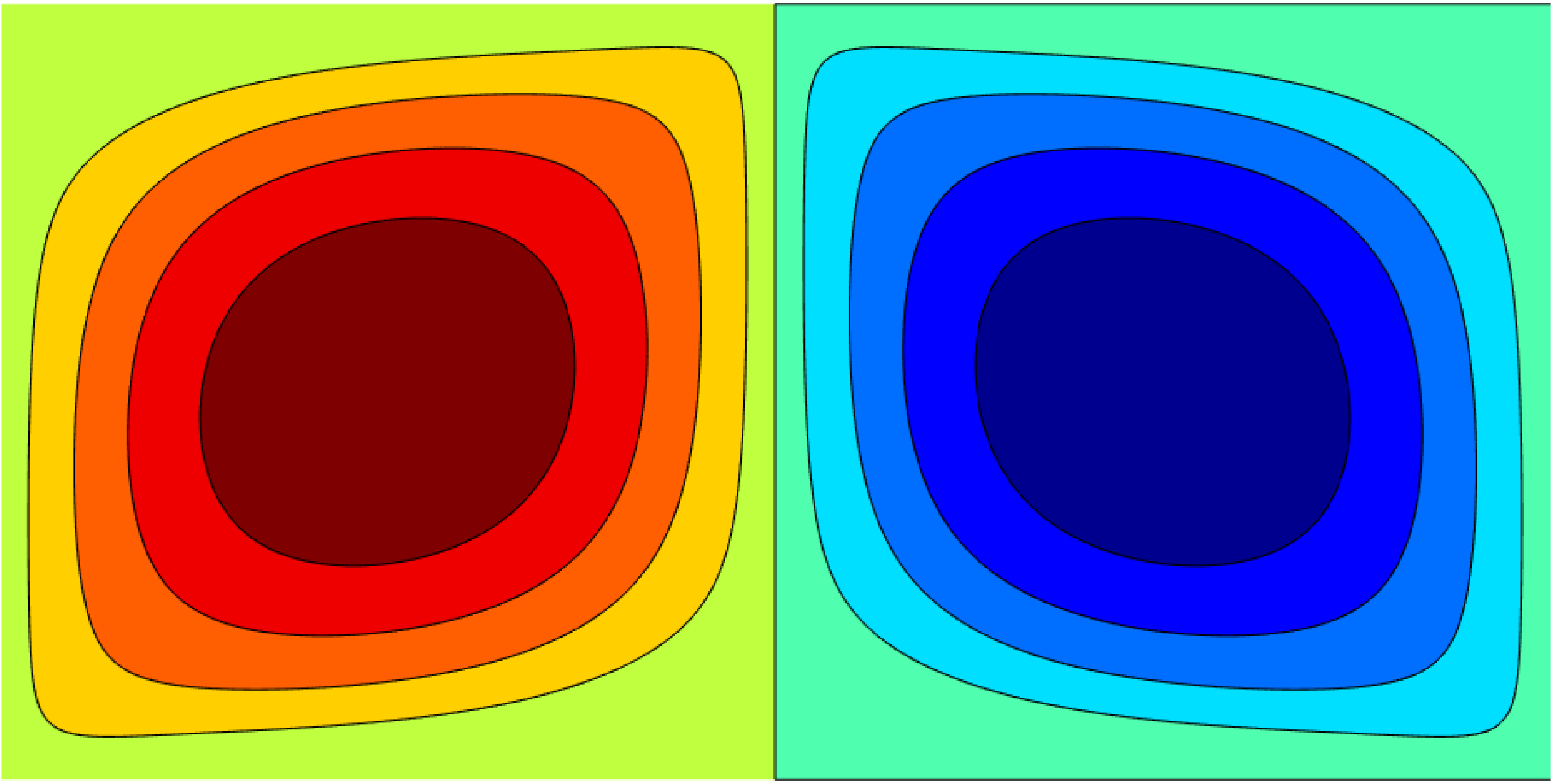
(1) Primary branch, $L/H = 2$, Streamfunction Ψ

$Ra = 20\,000\,000$



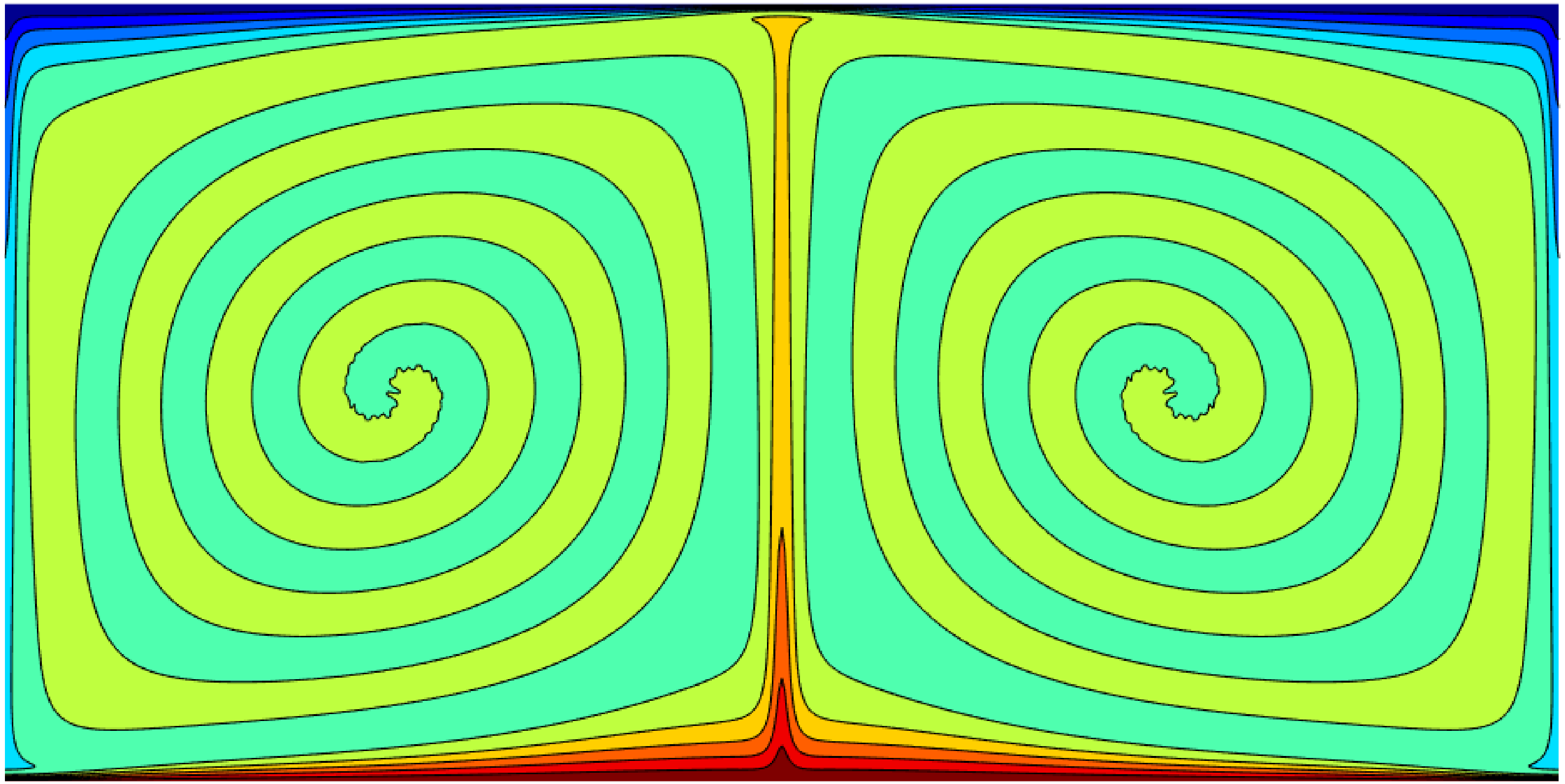
(1) Primary branch, $L/H = 2$, Streamfunction Ψ

$Ra = 40\,000\,000$

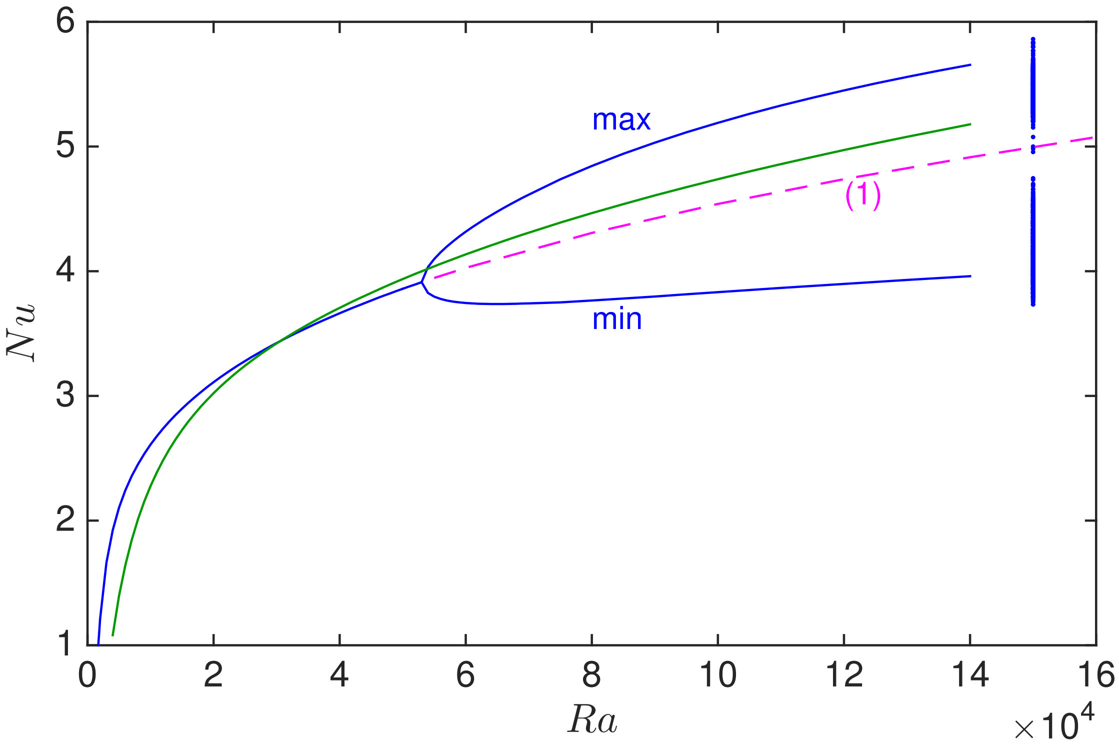


(1) Primary branch, $L/H = 2$, unstable for $Ra > 53\,000$

$Ra = 40\,000\,000$

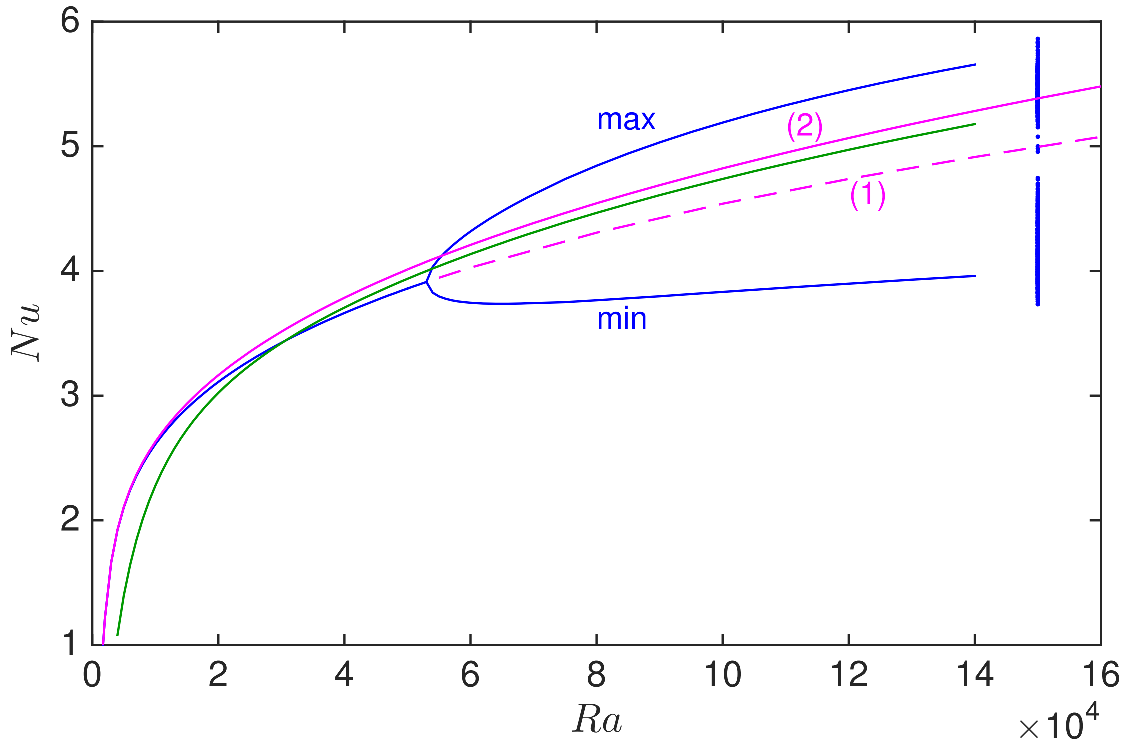


Continuation of unstable Primary (1)



Primary mode $\alpha \approx \pi/2$ (blue), Hopf bifurcation near *Ra* $\approx 53\,000$. Second mode $\alpha = \pi$ (green).

Continuation of Optimum Branch (2) now

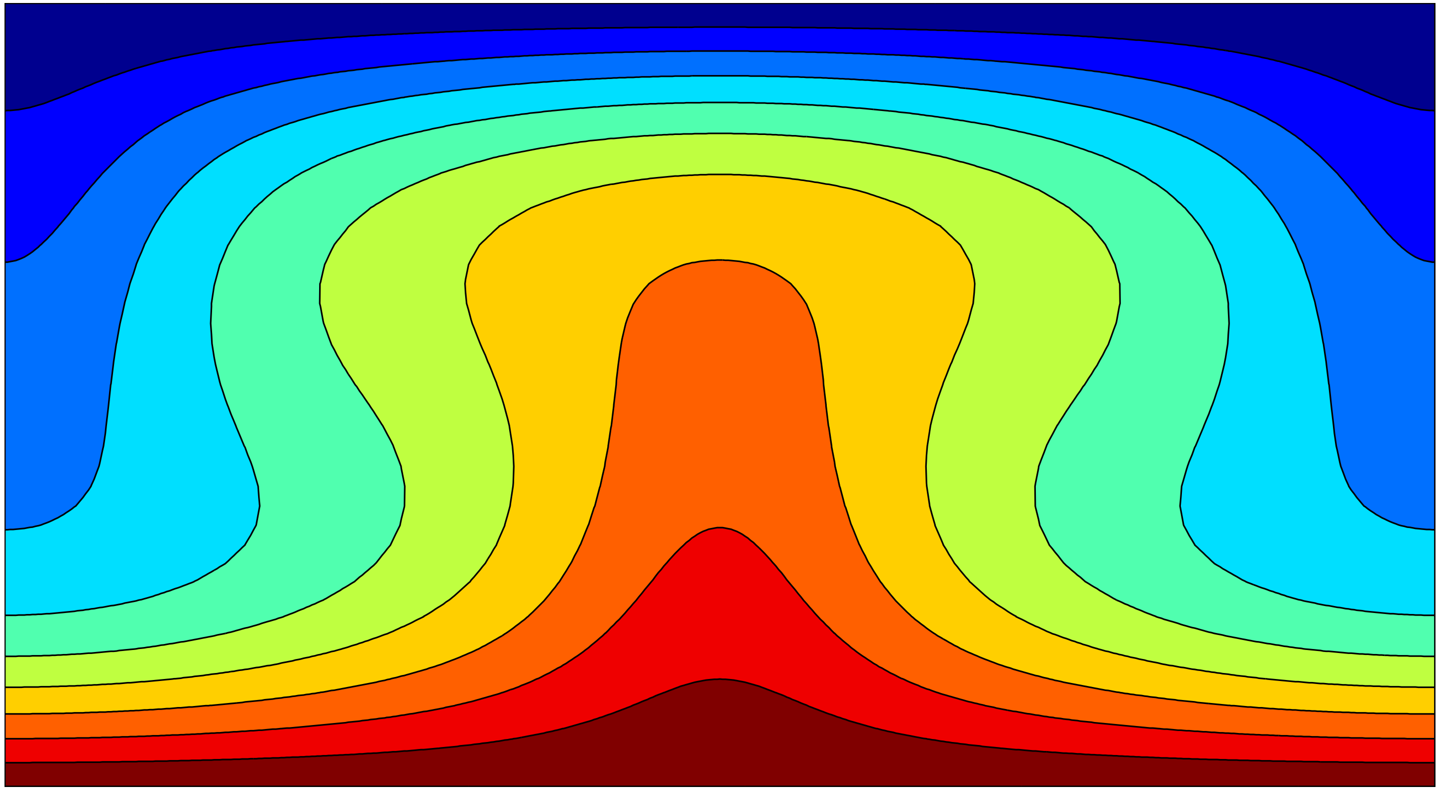


Primary mode $\alpha \approx \pi/2$ (blue), Hopf bifurcation near *Ra* $\approx$ 53 000.

Second mode $\alpha = \pi$ (green).

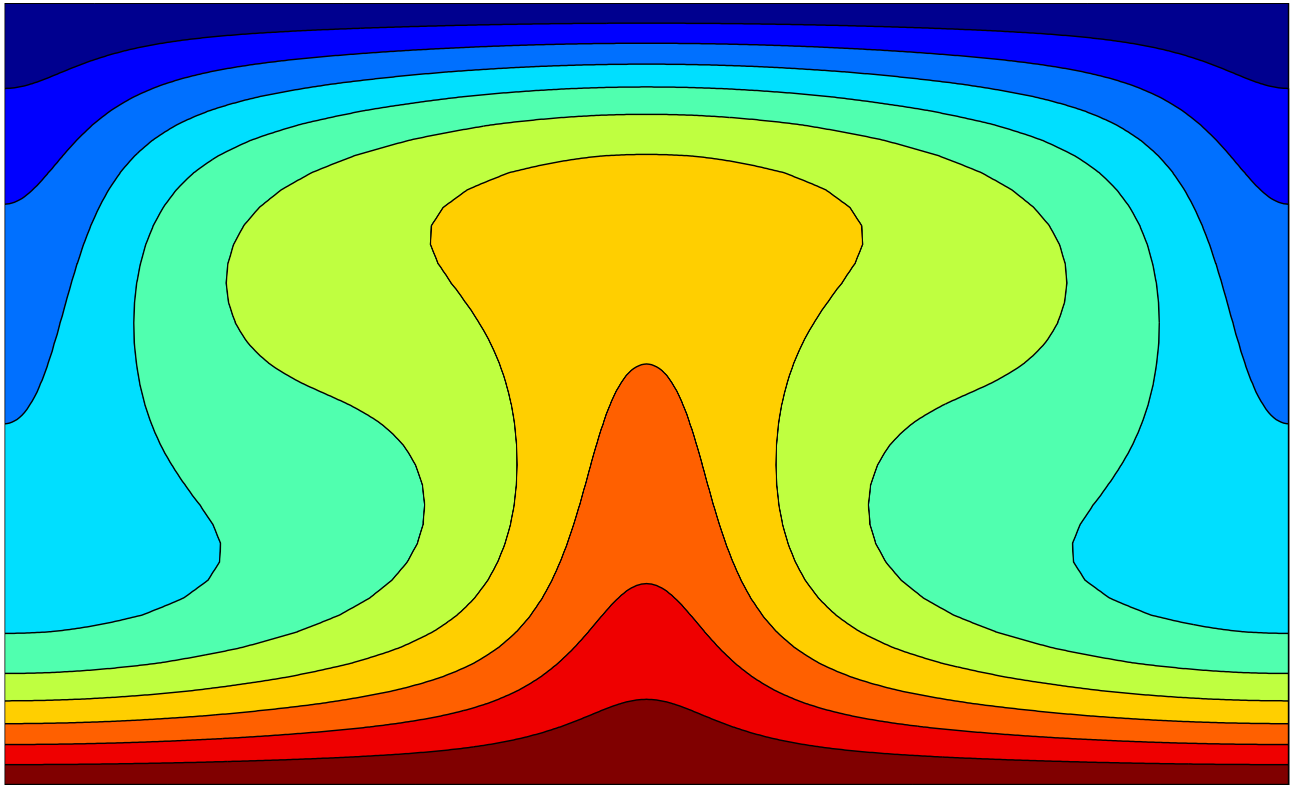
(2) Optimum branch: pick α to maximize Nu

$Ra = 6\,000$



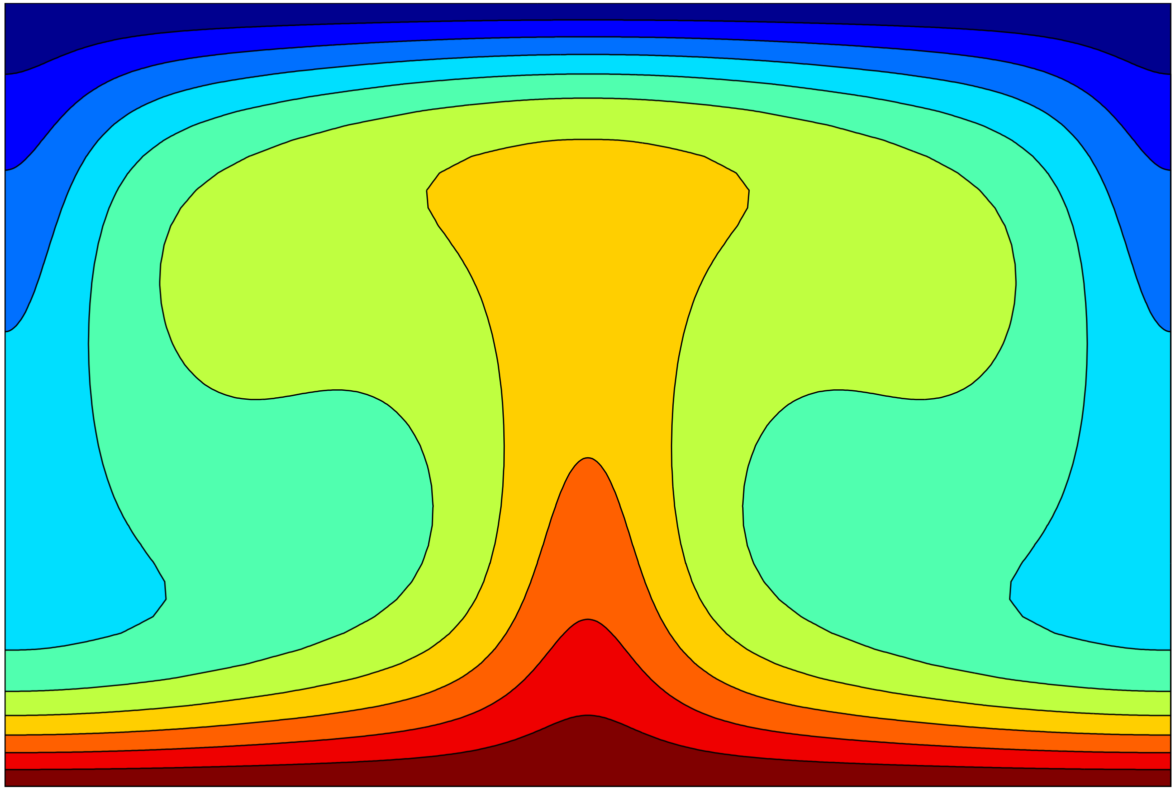
(2) Optimum branch: pick α to maximize Nu

$$Ra = 12\,000$$



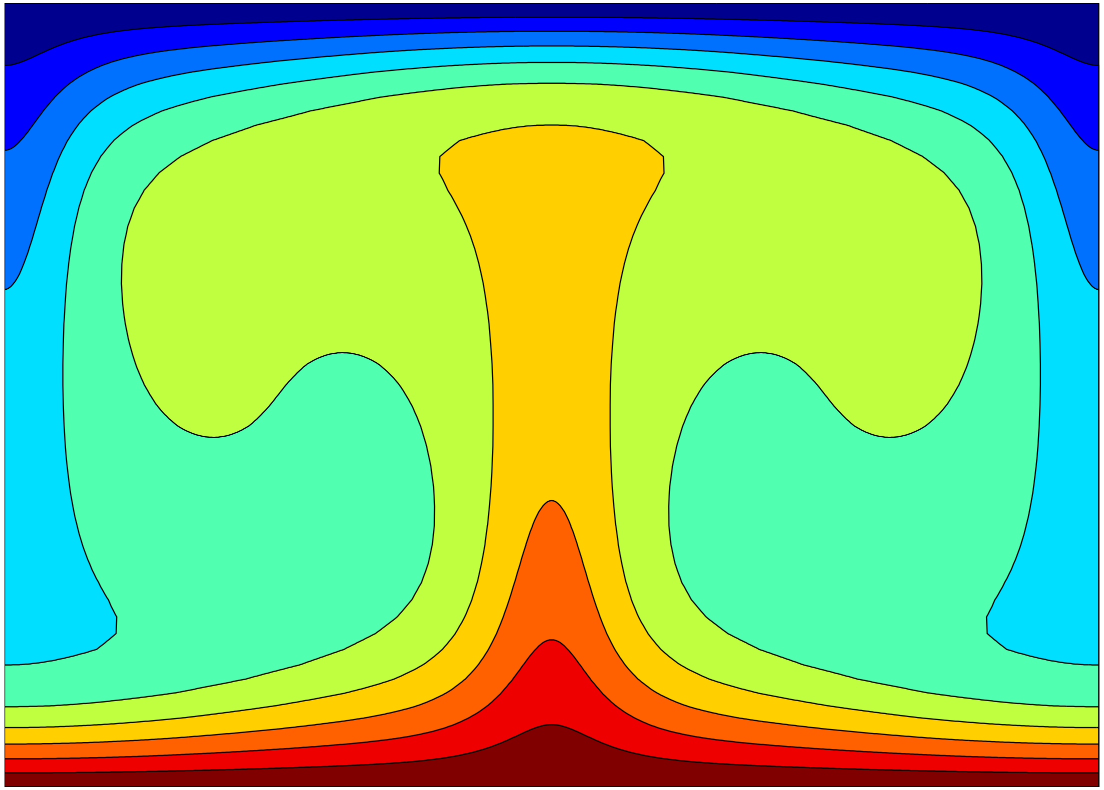
(2) Optimum branch: pick α to maximize Nu

$$Ra = 25\,000$$



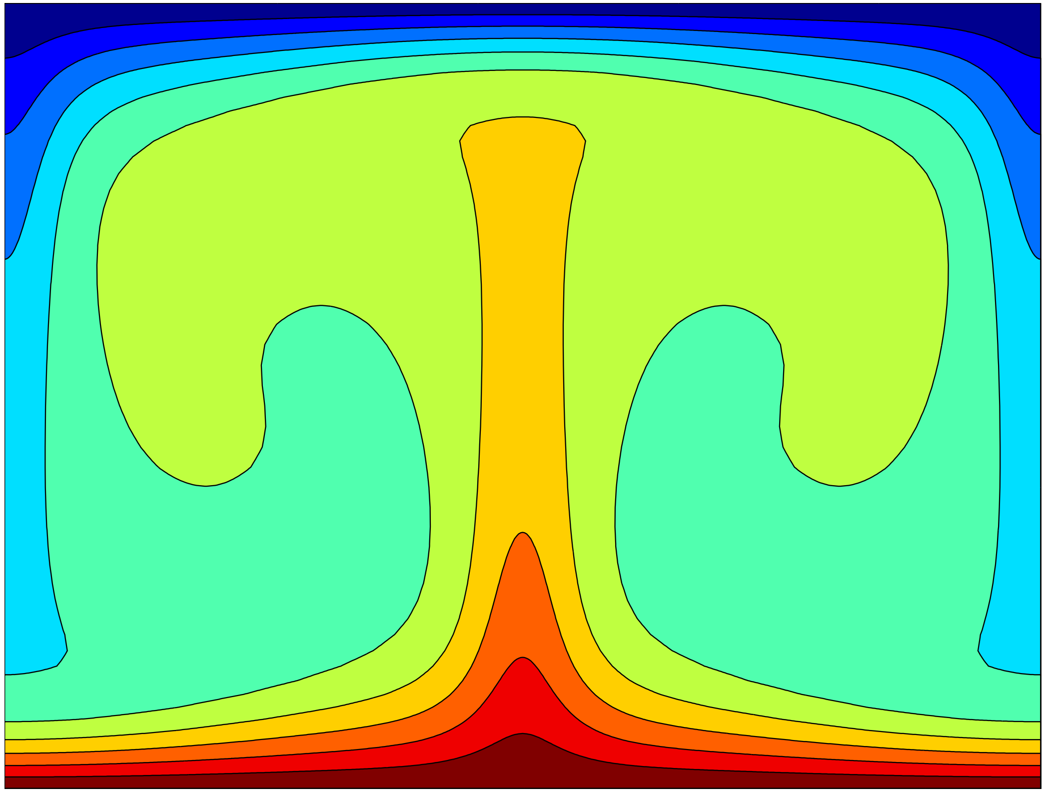
(2) Optimum branch: pick α to maximize Nu

$Ra = 50\,000$



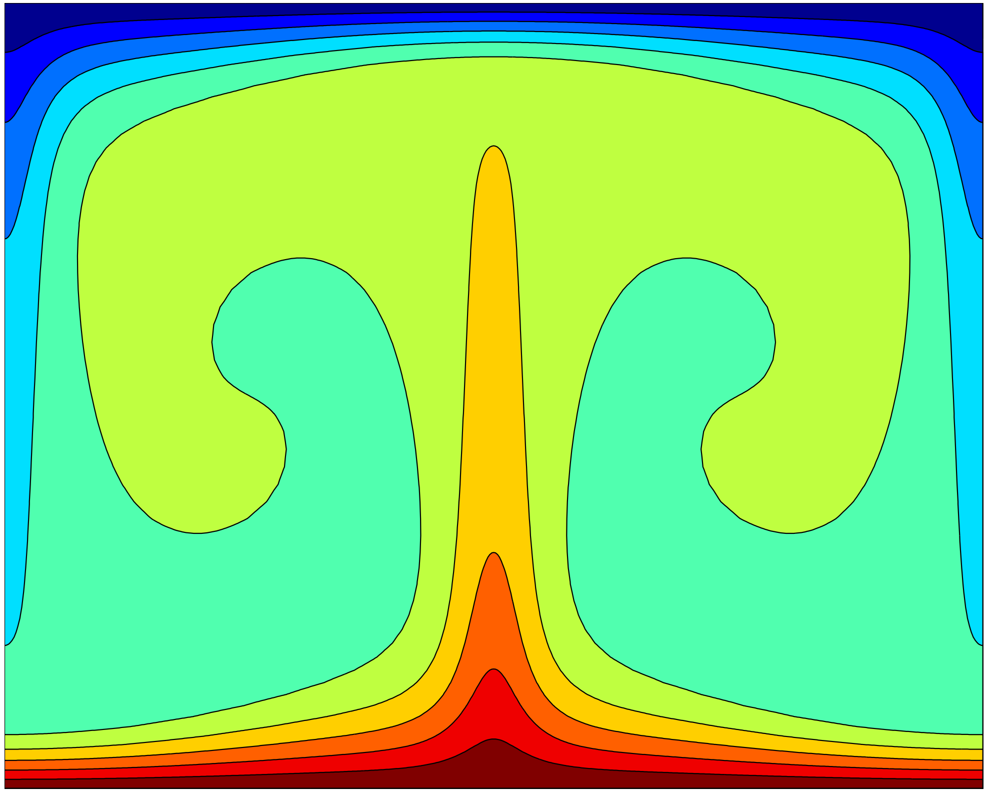
(2) Optimum branch: pick α to maximize Nu

$Ra = 100\,000$



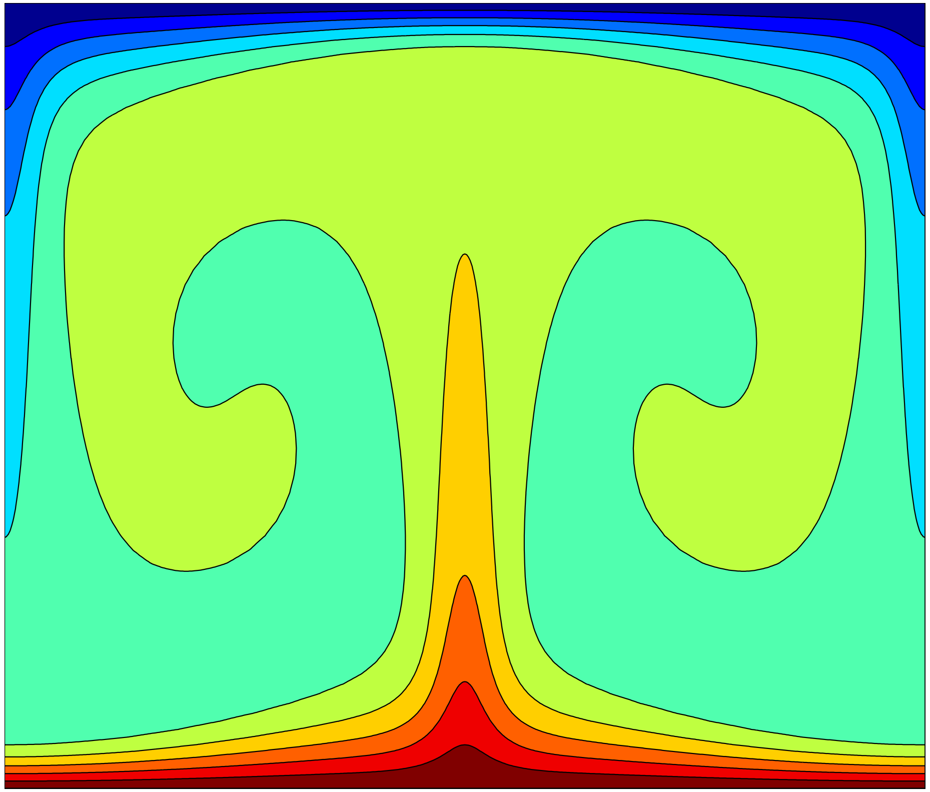
(2) Optimum branch: pick α to maximize Nu

$$Ra = 200\,000$$



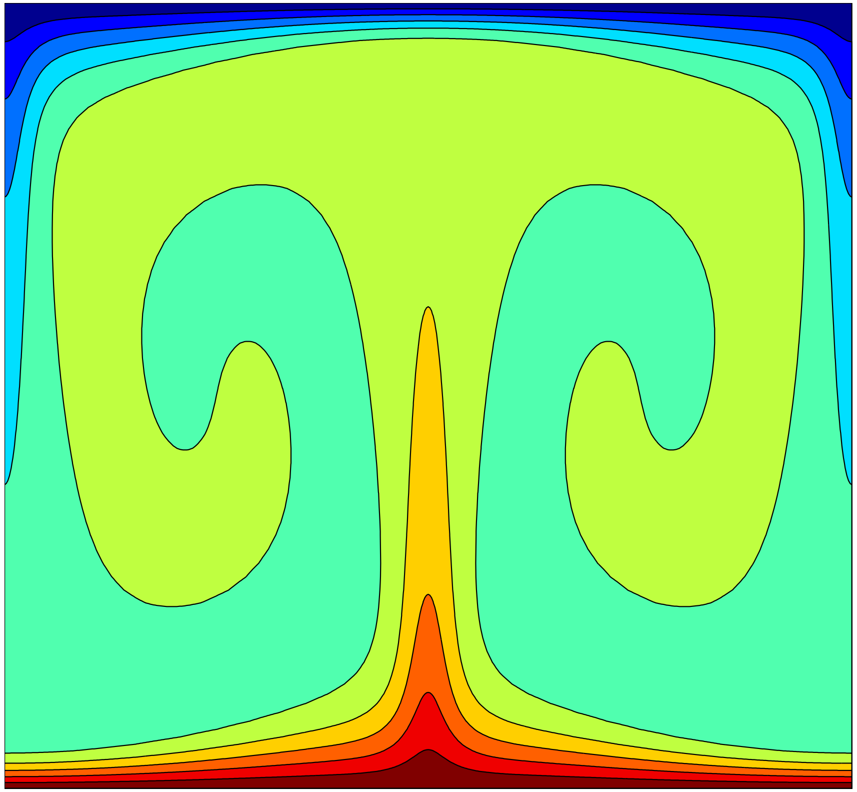
(2) Optimum branch: pick α to maximize Nu

$Ra = 400\,000$



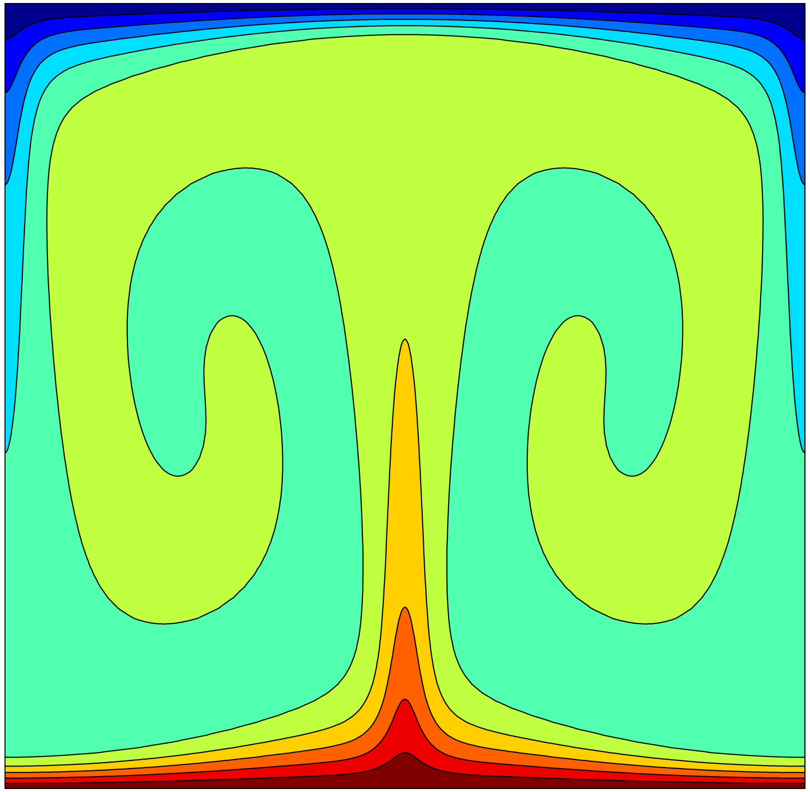
(2) Optimum branch: pick α to maximize Nu

$Ra = 800\,000$



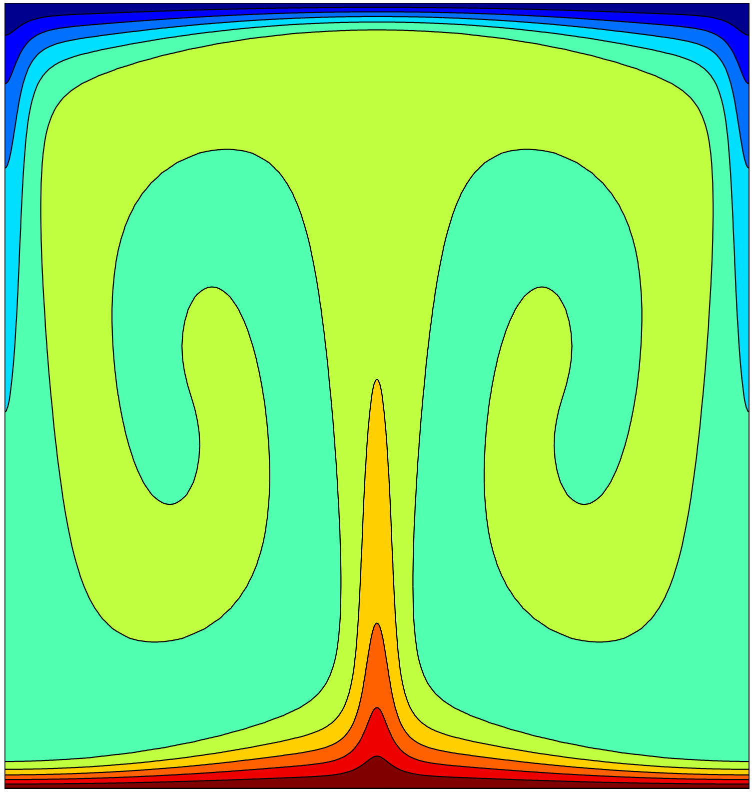
(2) Optimum branch: pick α to maximize Nu

$Ra = 1\,200\,000$



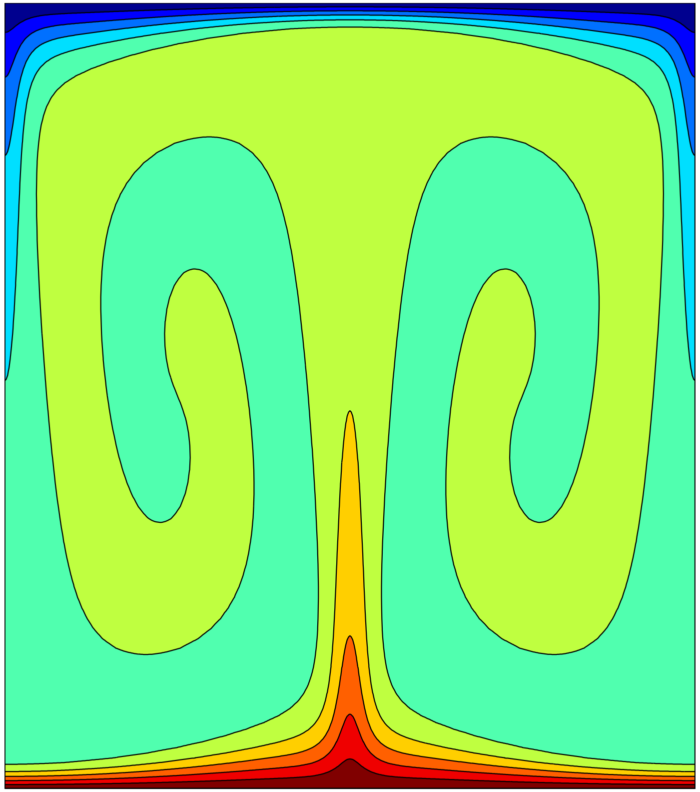
(2) Optimum branch: pick α to maximize Nu

$$Ra = 2\,000\,000$$



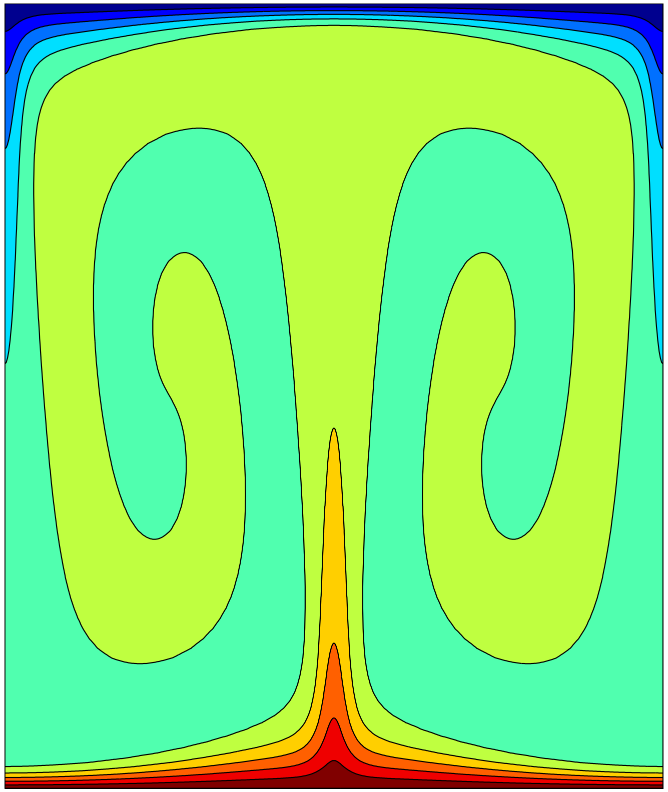
(2) Optimum branch: pick α to maximize Nu

$Ra = 3\,000\,000$



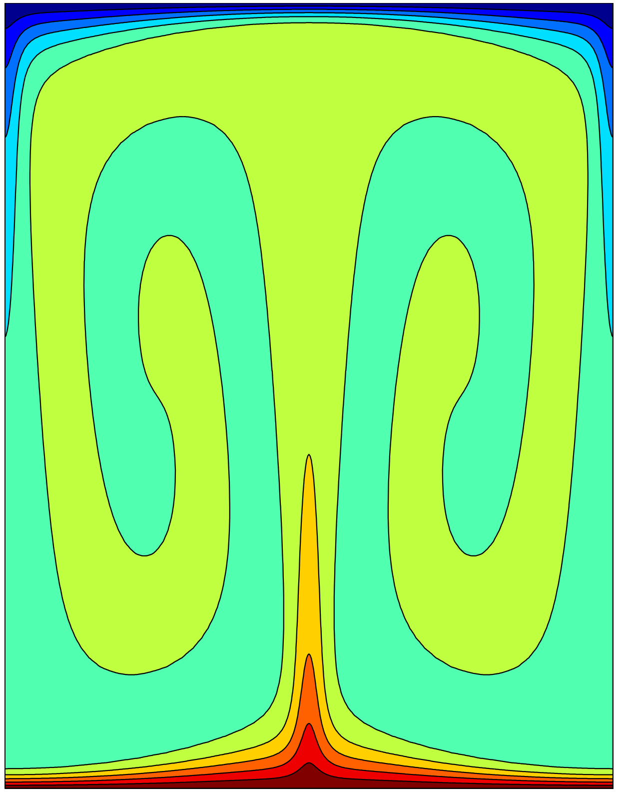
(2) Optimum branch: pick α to maximize Nu

$Ra = 4\,000\,000$



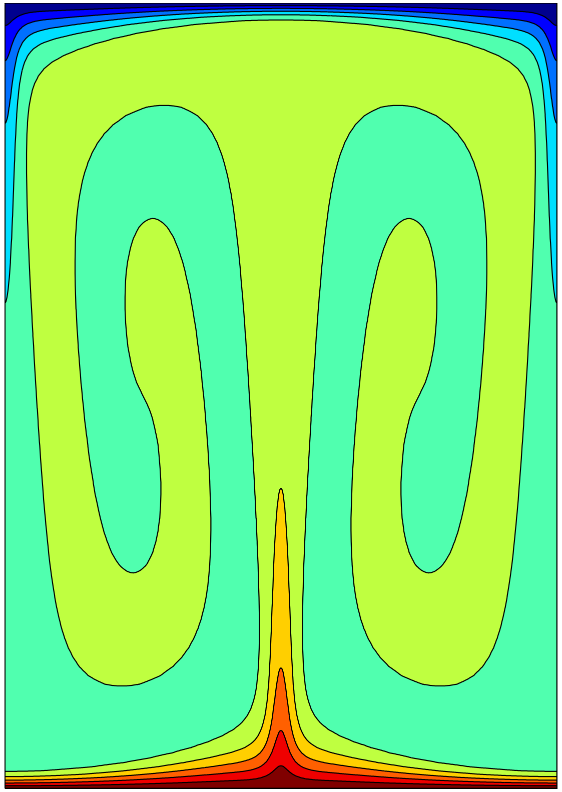
(2) Optimum branch: pick α to maximize Nu

$$Ra = 6\,000\,000$$



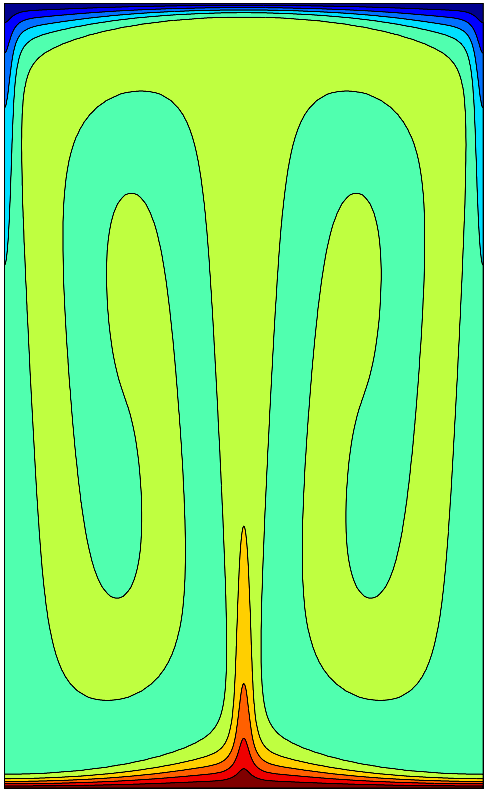
(2) Optimum branch: pick α to maximize Nu

$Ra = 10\,000\,000$



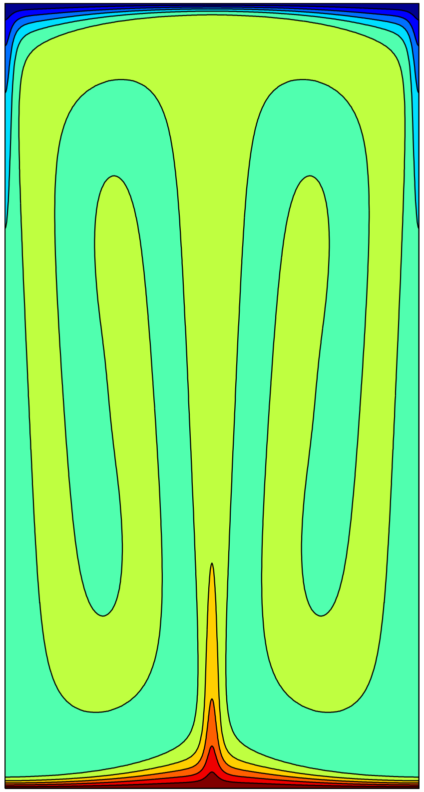
(2) Optimum branch: pick α to maximize Nu

$Ra = 20\,000\,000$



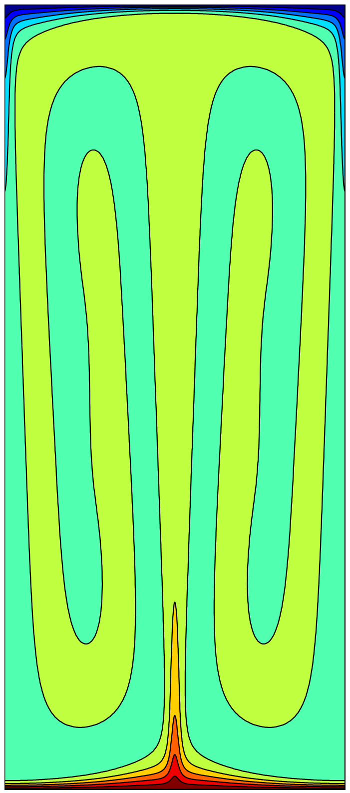
(2) Optimum branch: pick α to maximize Nu

$Ra = 40\,000\,000$



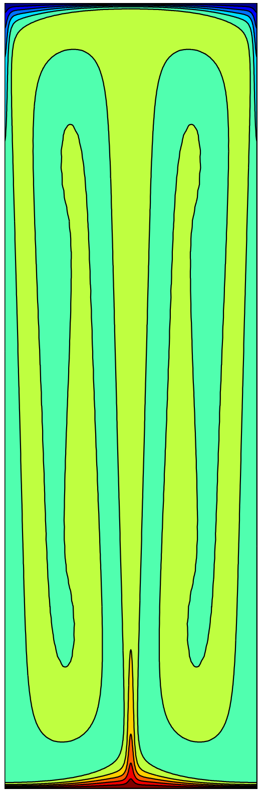
(2) Optimum branch: pick α to maximize Nu

$Ra = 100\,000\,000$



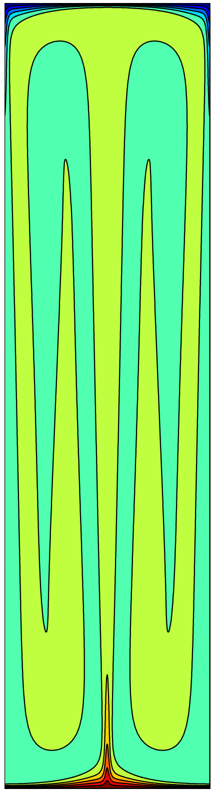
(2) Optimum branch: pick α to maximize Nu

$$Ra = 400\,000\,000$$



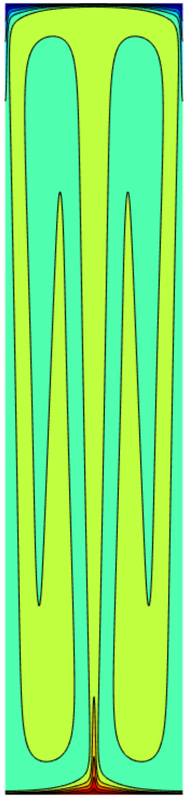
(2) Optimum branch: pick α to maximize Nu

$$Ra = 1\,000\,000\,000$$



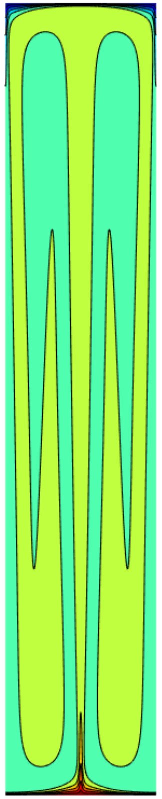
(2) Optimum branch: pick α to maximize Nu

$$Ra = 2\,000\,000\,000$$



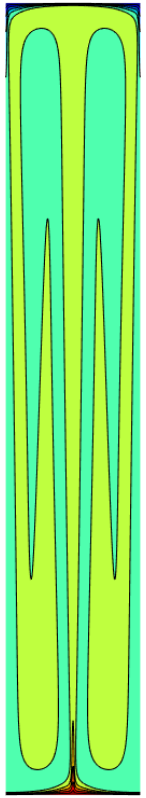
(2) Optimum branch: pick α to maximize Nu

$$Ra = 4\,000\,000\,000$$

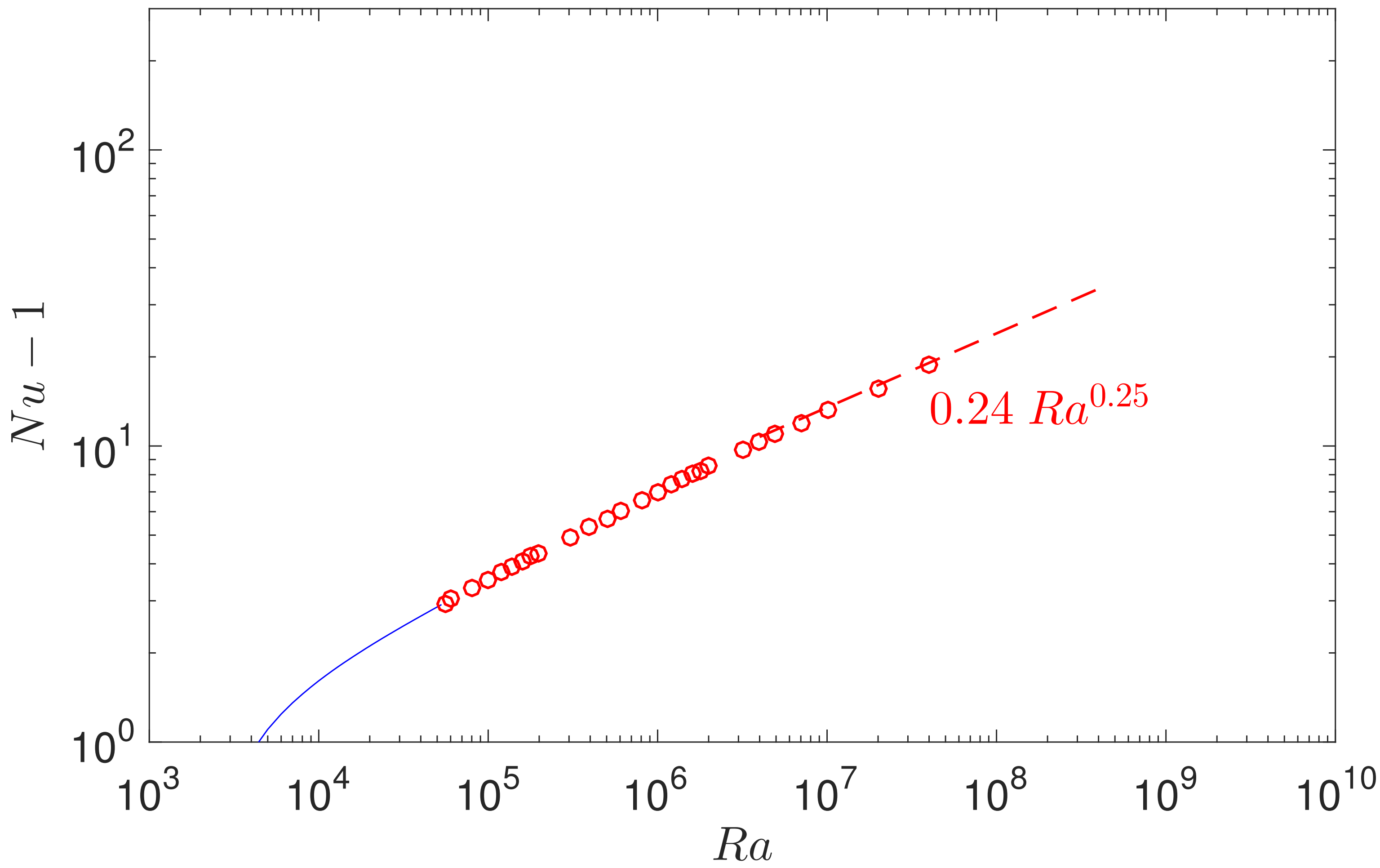


(2) Optimum branch: pick α to maximize Nu

$Ra = 7\,000\,000\,000$

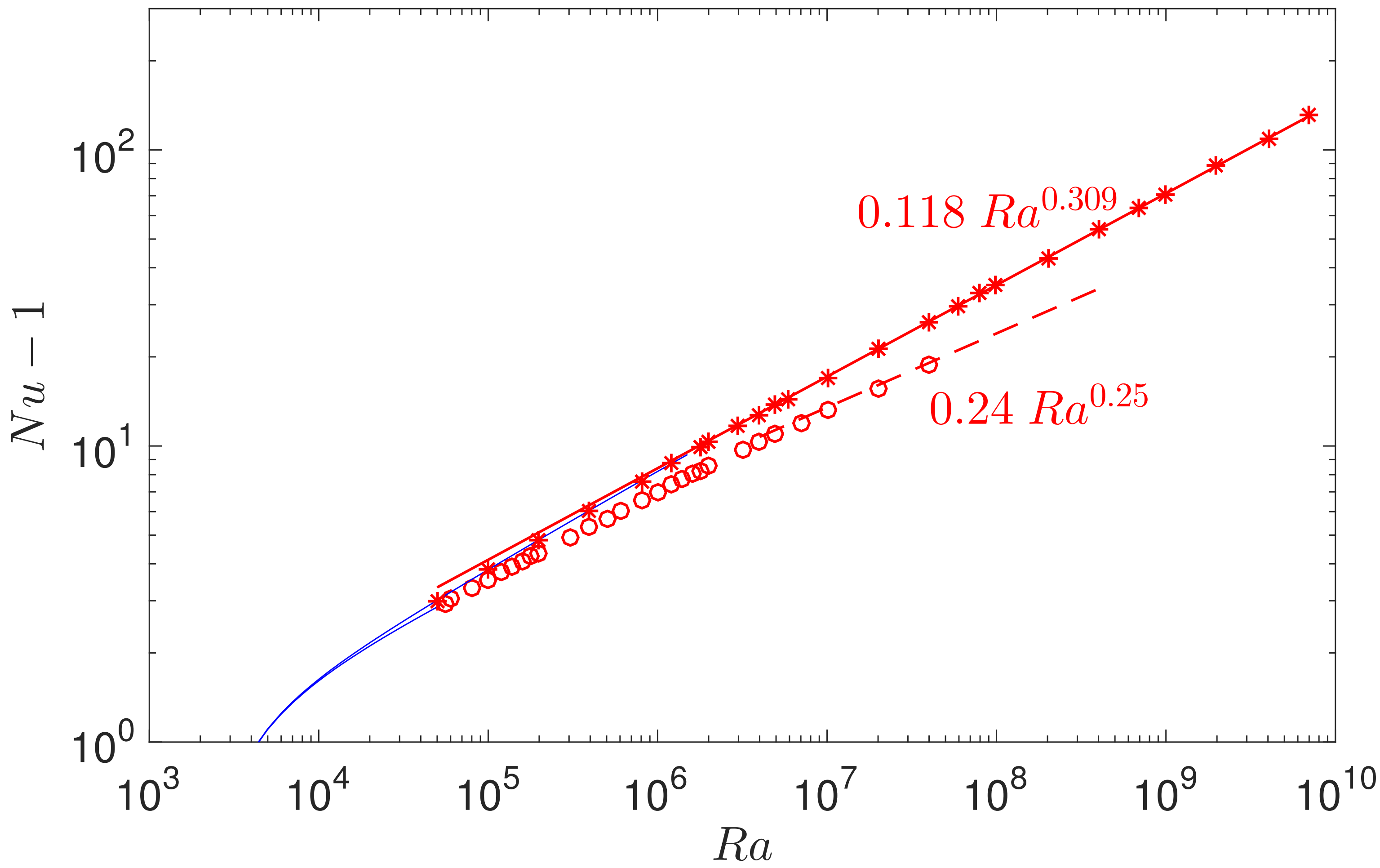


$Nu(Ra)$: primary



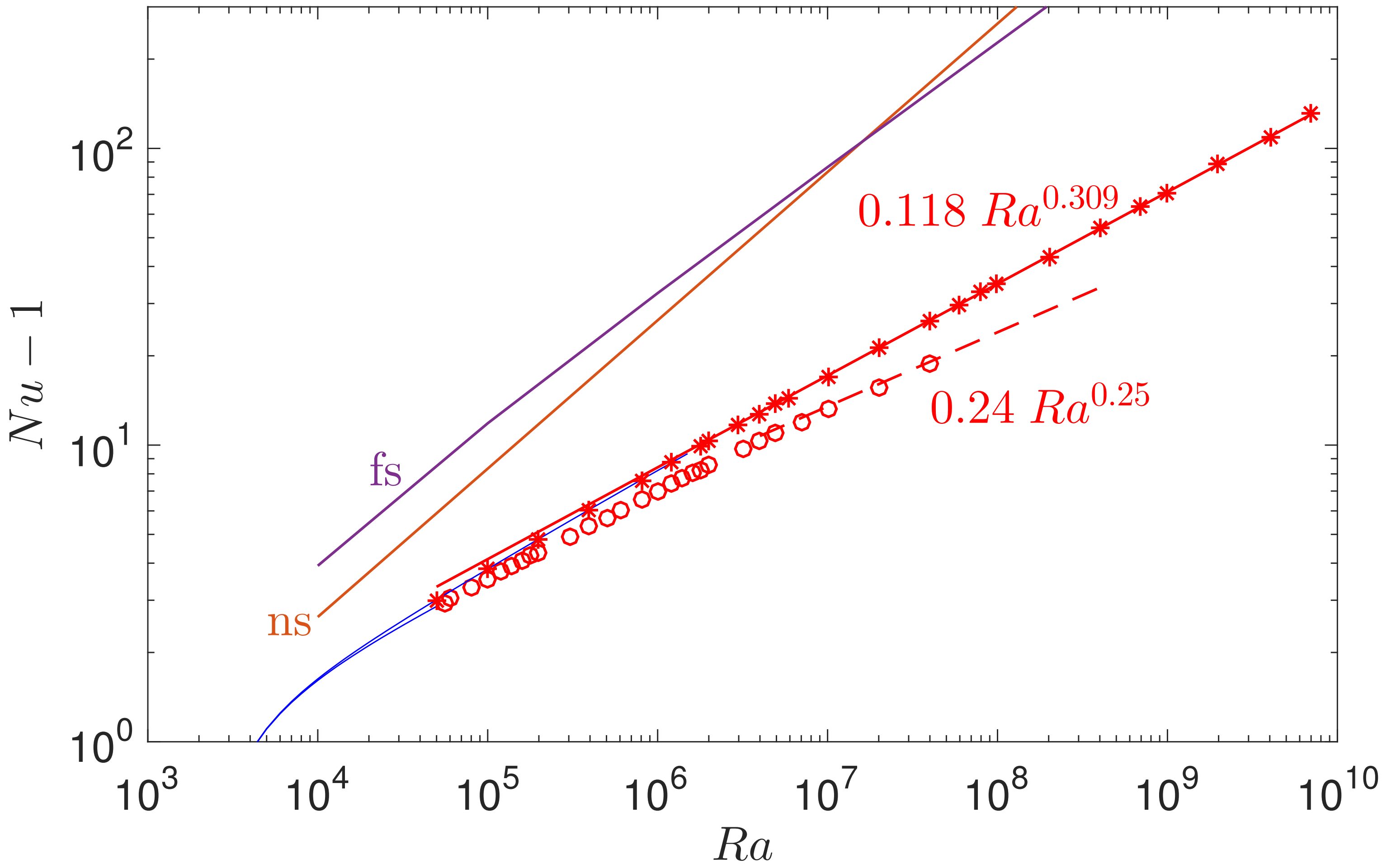
Waleffe, Boonkasame, Smith, *Phys Fluids 2015* + new results

$Nu(Ra)$: primary, optimum



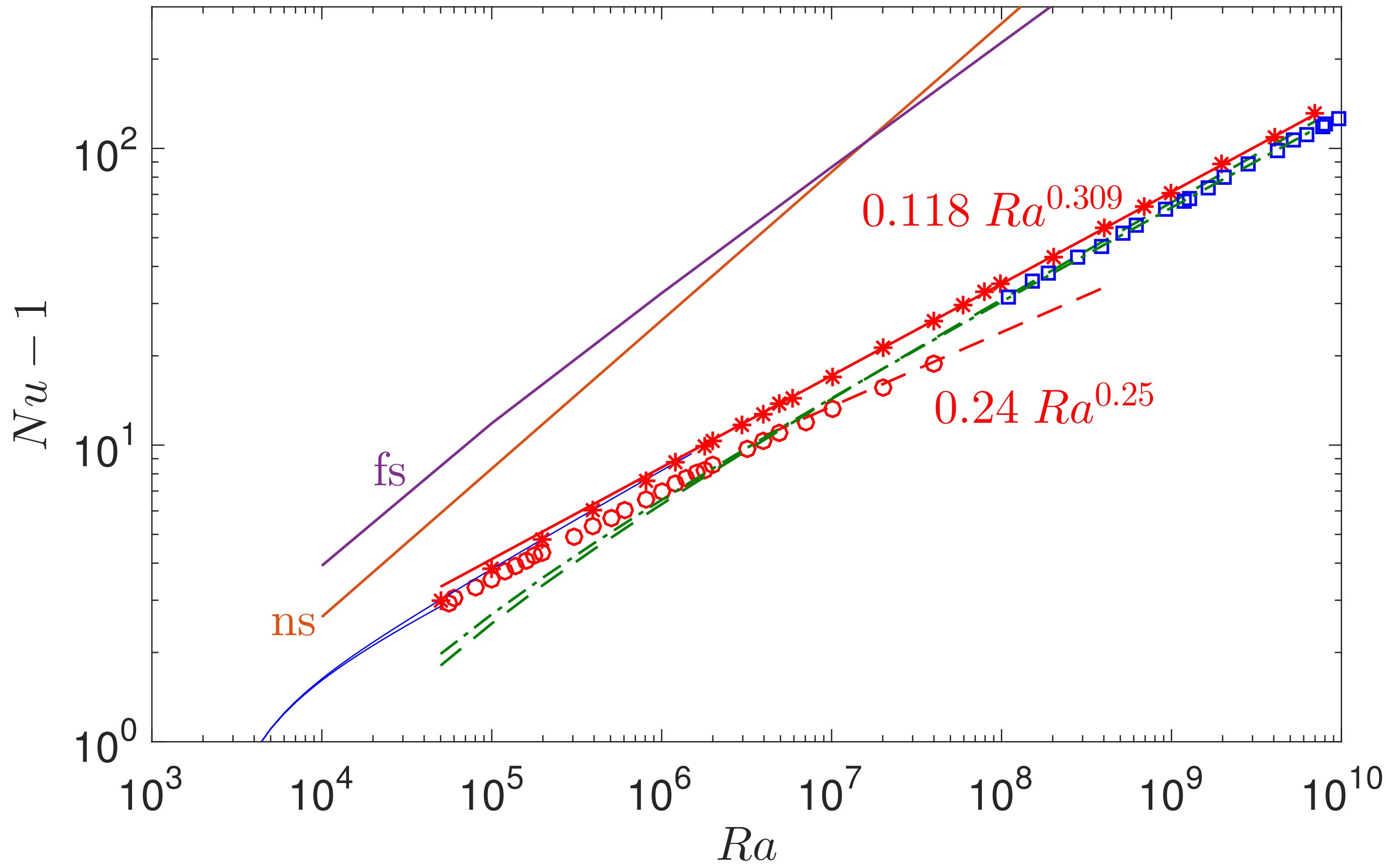
Waleffe, Boonkasame, Smith, *Phys Fluids 2015* + new results

$Nu(Ra)$: primary, optimum, Upper Bounds



Waleffe, Boonkasame, Smith, *Phys Fluids 2015* + new results

$Nu(Ra)$: primary, optimum, Upper Bounds, Exp. Data



Waleffe, Boonkasame, Smith, *Phys Fluids 2015* + new results

$Nu(Ra)$ experimental data sets

- $Nu \sim 0.088\, Ra^{0.32}$ fit of 3D turbulent data for $D/H = 1/2$
  (Niemela, Sreenivasan, et al., 2000-2006)
- $Nu \sim 0.105\, Ra^{0.31}$ fit of 3D turbulent data for $D/H = 1/2$
  (He, Funfschilling, Nobach, Bodenschatz, Ahlers, 2012)
- $Nu(Ra)$ 3D turbulent data for $D/H = 4$
  (Niemela and Sreenivasan 2006)
- ...

Niemela and Sreenivasan 2006

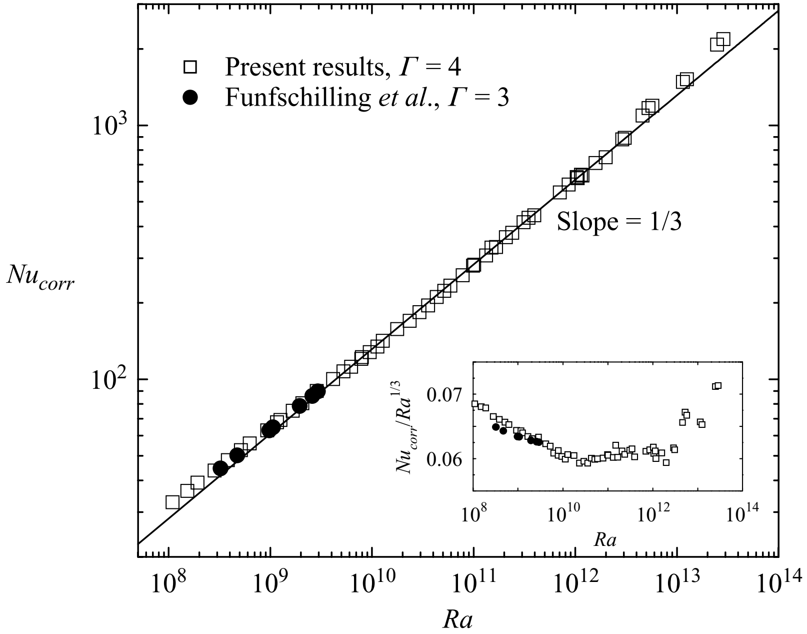
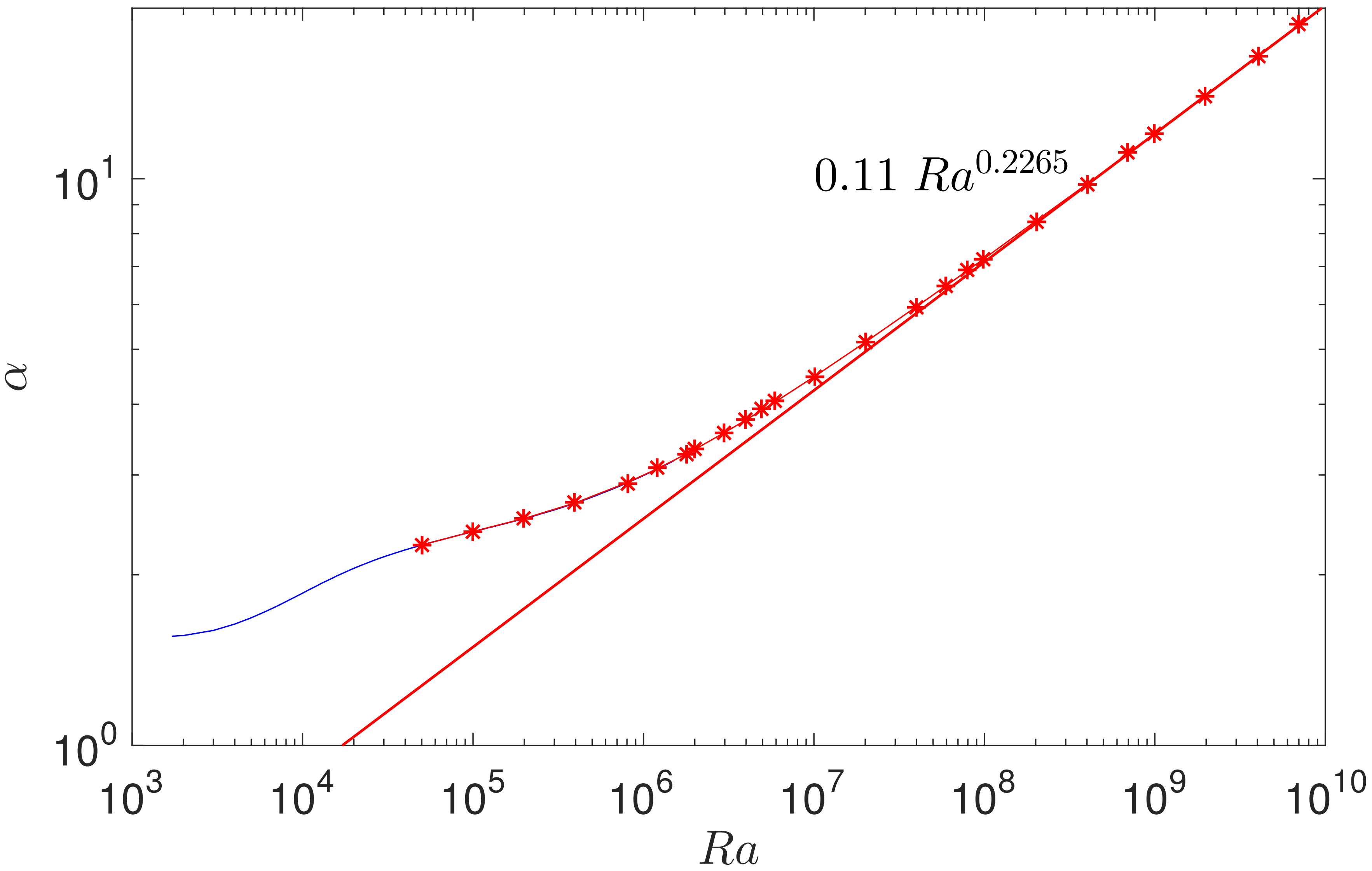


FIGURE 3. Nu_{corr} versus Ra for the present data ($\Gamma = 4$), adjusted for the effects of sidewall and horizontal plates. Also shown are recent results of Funfschilling *et al.* (2005) for aspect ratio 3, similarly corrected. The inset shows the same Nu_{corr} data normalized by $Ra^{1/3}$.

of $10^{10} < Ra < 10^{12}$, with the corrections described above raising the slope over the uncorrected data, which have a slope of more precisely $1/3$. For the data falling in the range $10^8 < Ra < 10^{10}$ the local log–log slope is nearly constant giving an exponent of 0.31. Both the constancy of the log–log slope with increasing Ra and its numerical value are in good agreement with predictions of Grossmann & Lohse (2002) in this range of Ra and for unity Pr (see their figure 4*b*). While the theory also predicts a saturation of the local exponent to $1/3$ at higher Ra, it occurs more slowly than is observed here. For comparison, we also show recent results of Funfschilling *et al.* (2005) also corrected for sidewall and end-plate effects. These data were obtained for $\Gamma = 3$ and $Pr = 4.38$, and accessed a limited range of Ra.

Horizontal wavenumber α vs. *Ra*, optimum

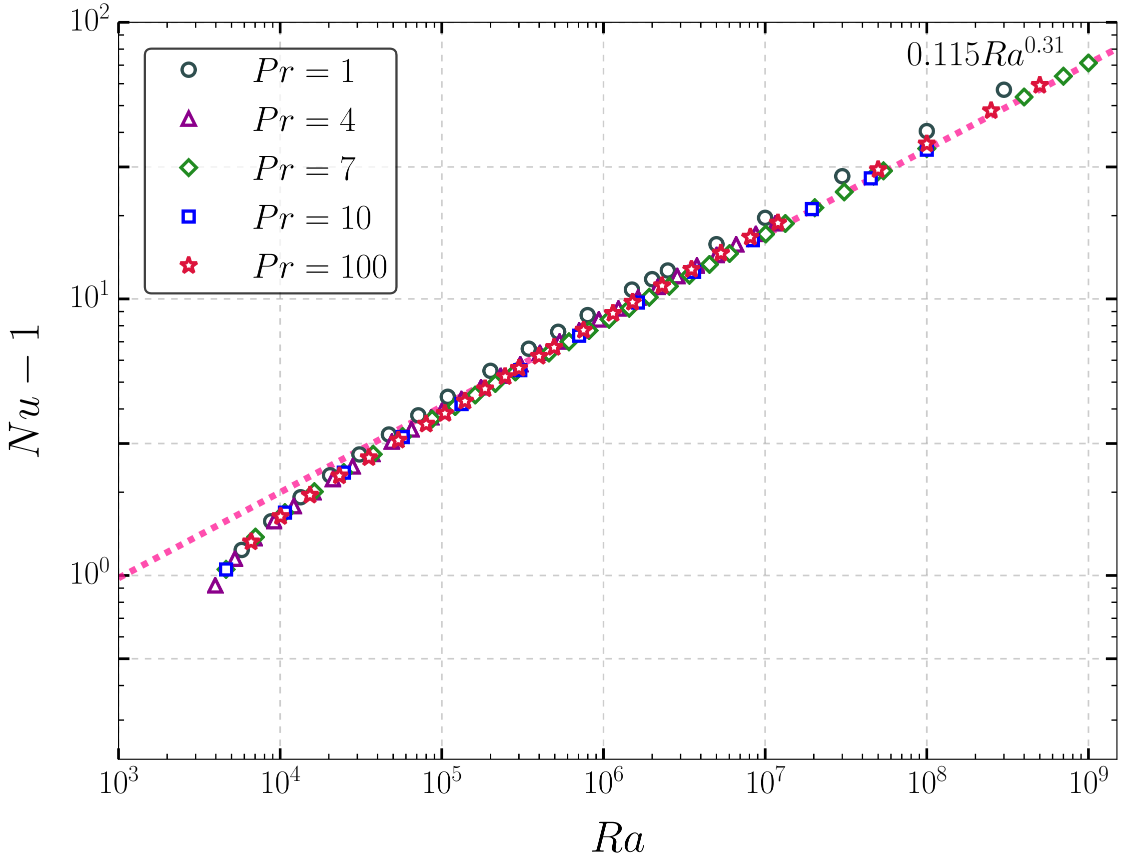


Wobble in $\alpha_{opt}(Ra)$ why?

Wobble in $\alpha_{opt}(Ra)$ why?

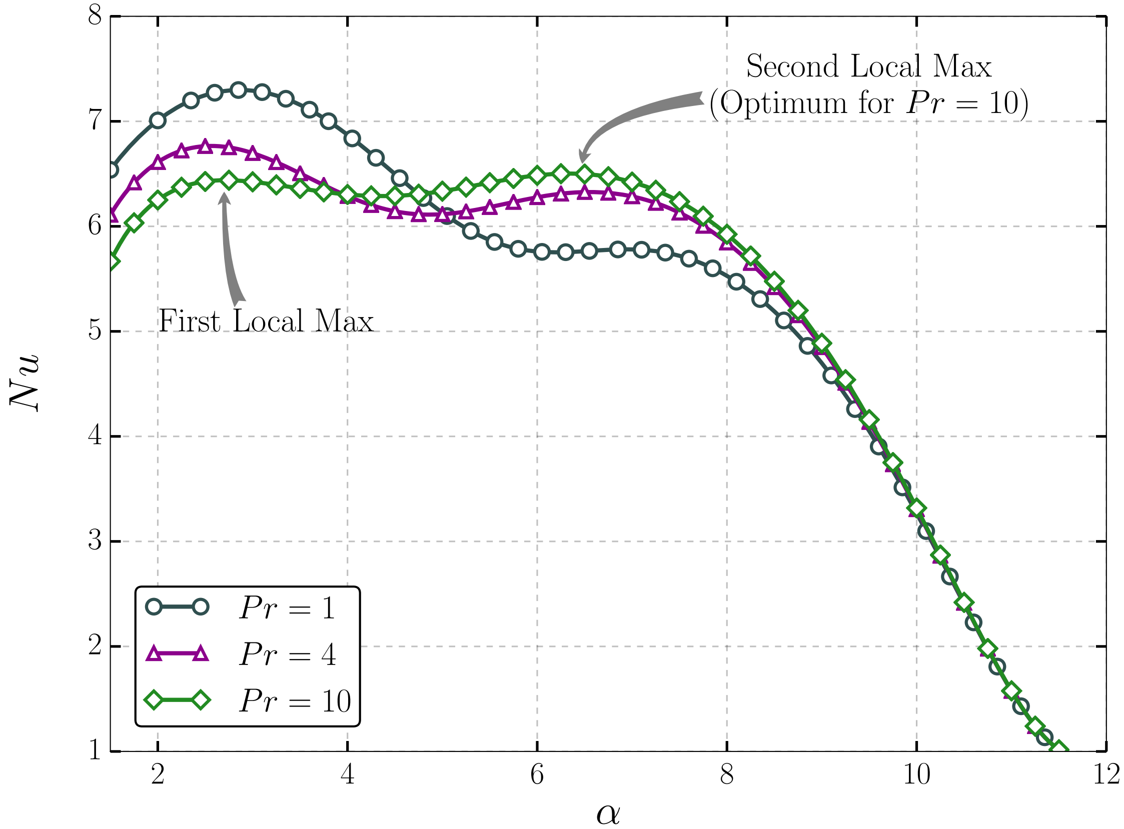
wrapping up of spiral structure

Varying *Pr* little effect on optimum *Nu*(*Ra*)



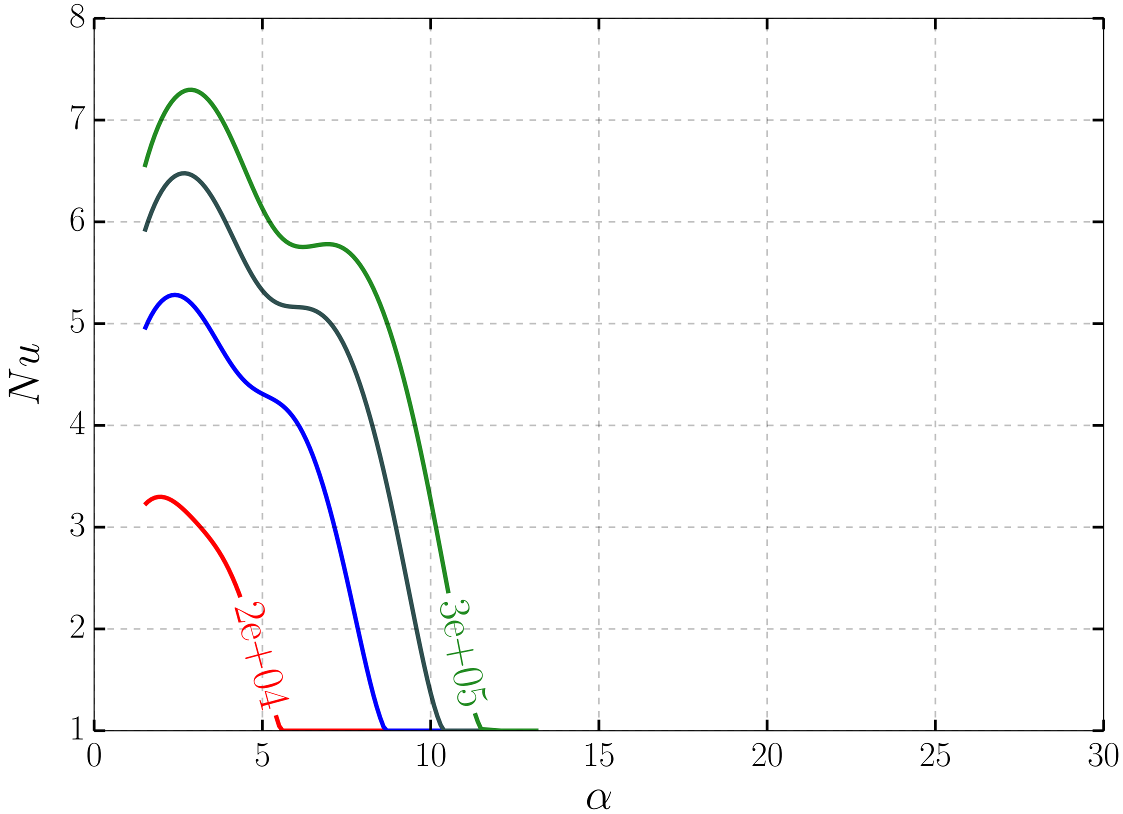
Sondak, Smith, Waleffe *JFM 2015*

Varying Pr but multiple local maxima! ($Ra = 3\ 10^5$)

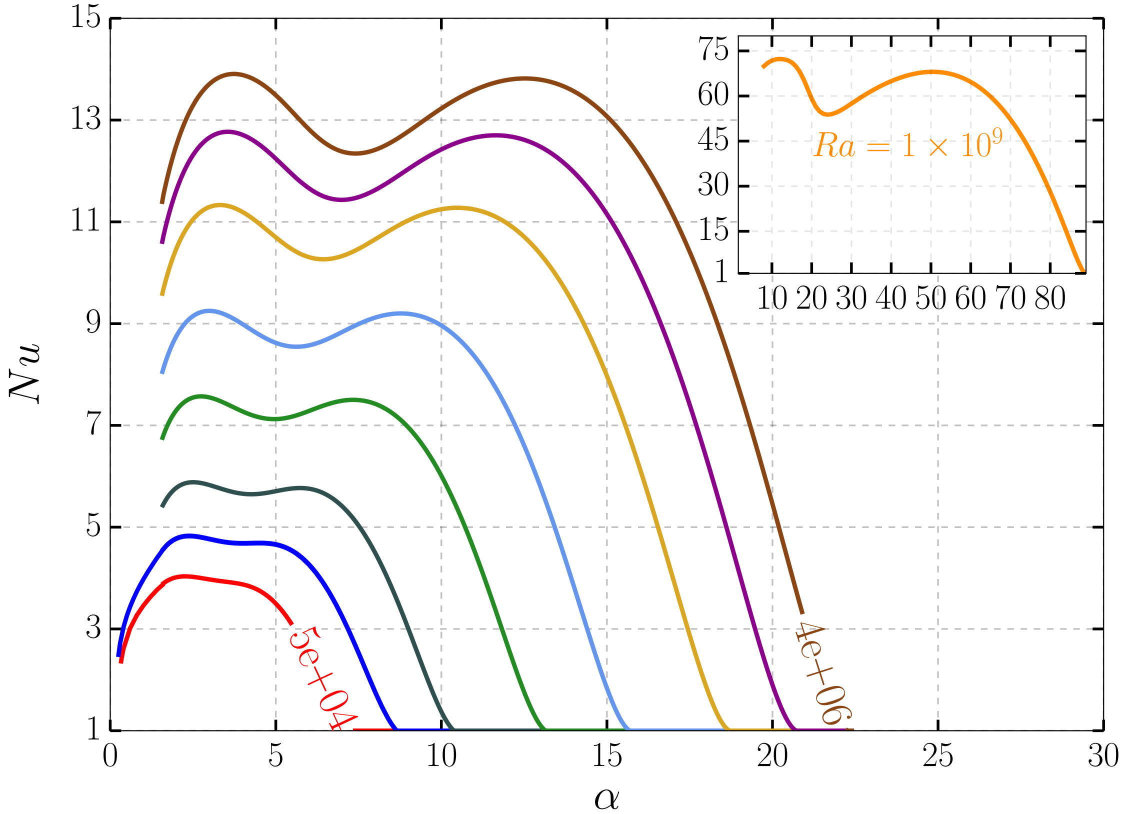


Sondak, Smith, Waleffe *JFM 2015*

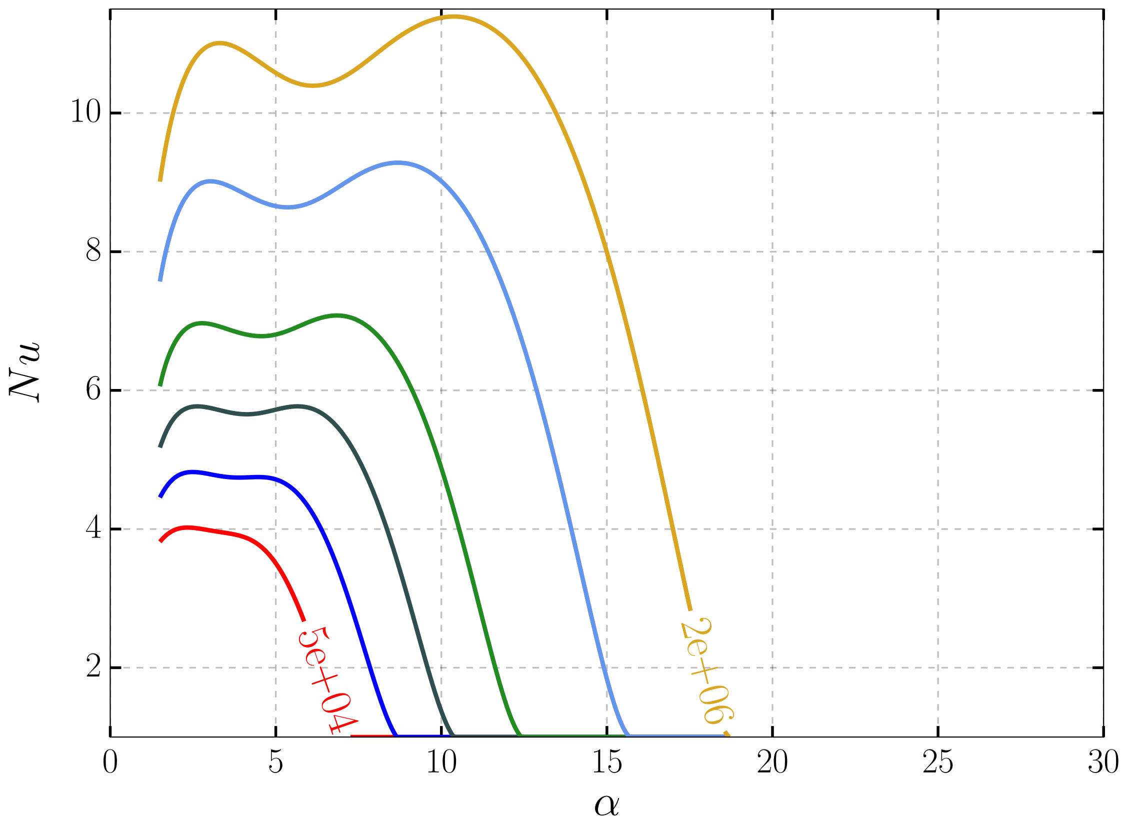
$Nu(\alpha, Ra)$, $Pr = 1$ 2nd max subdominant



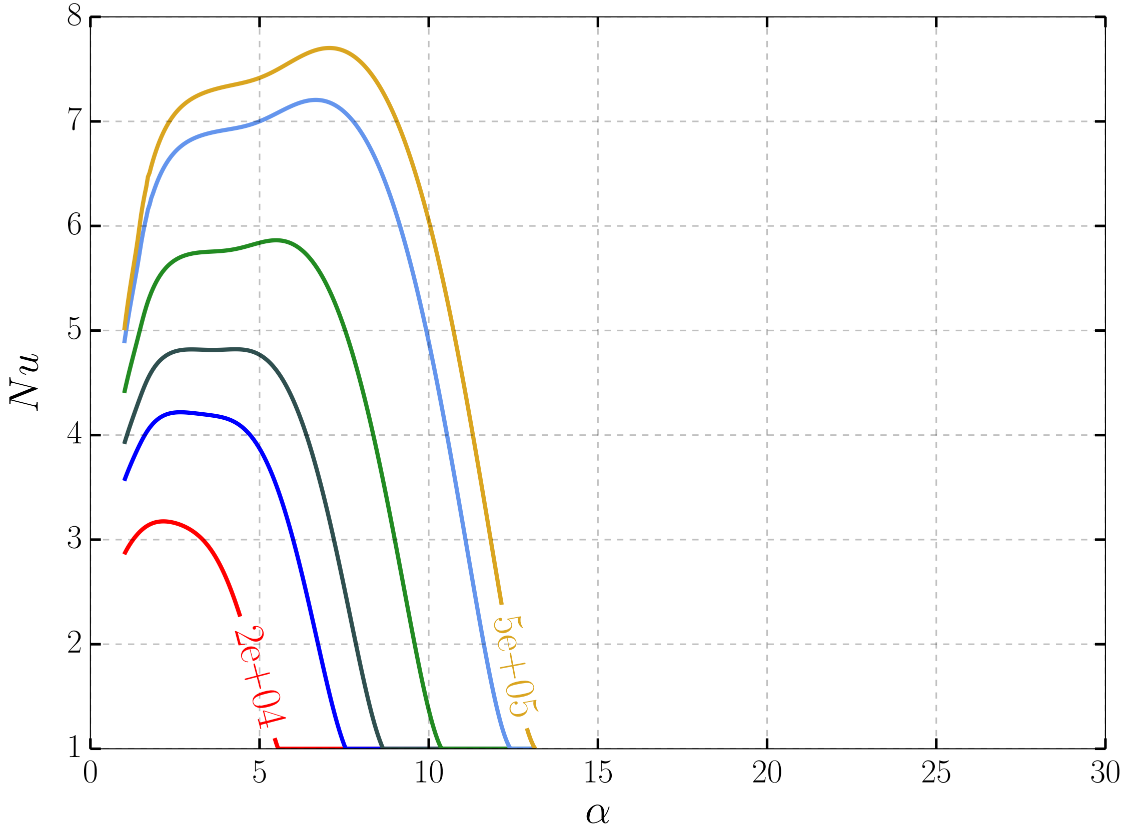
$Nu(\alpha, Ra)$, $Pr = 7$ 2nd max *almost* takes over



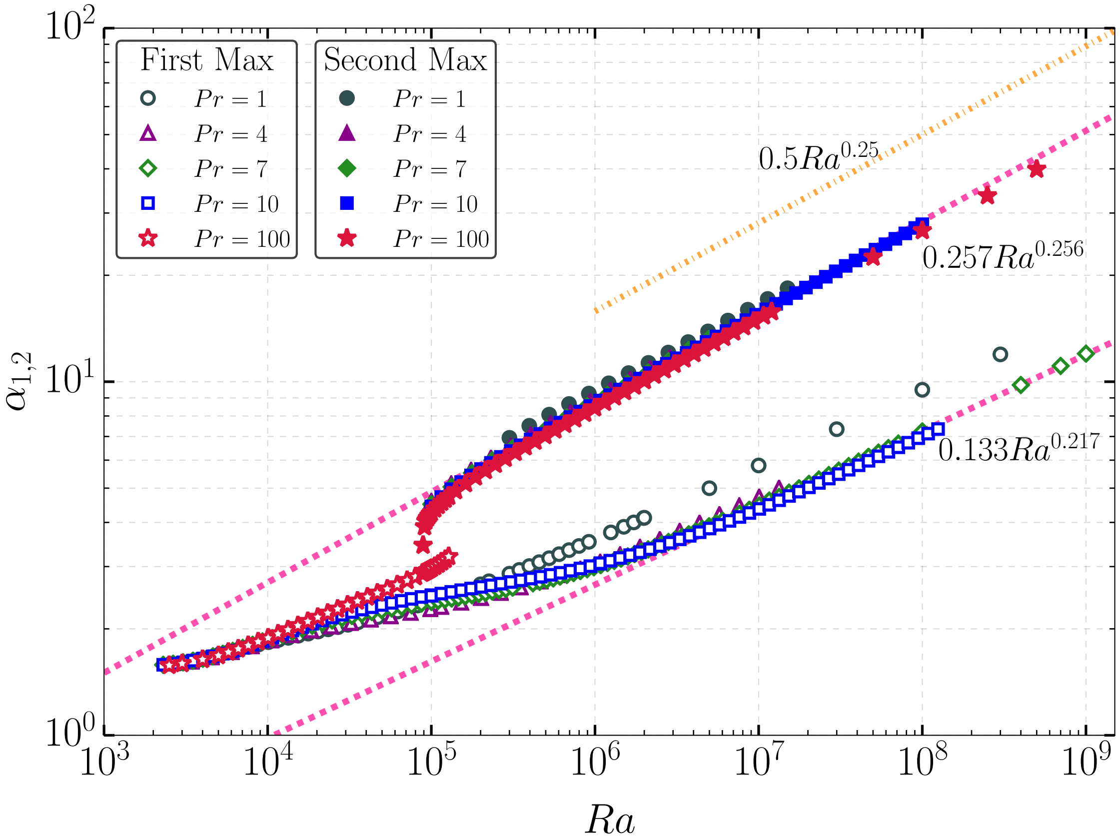
$Nu(\alpha, Ra)$, $Pr = 10$ 2nd max takes over



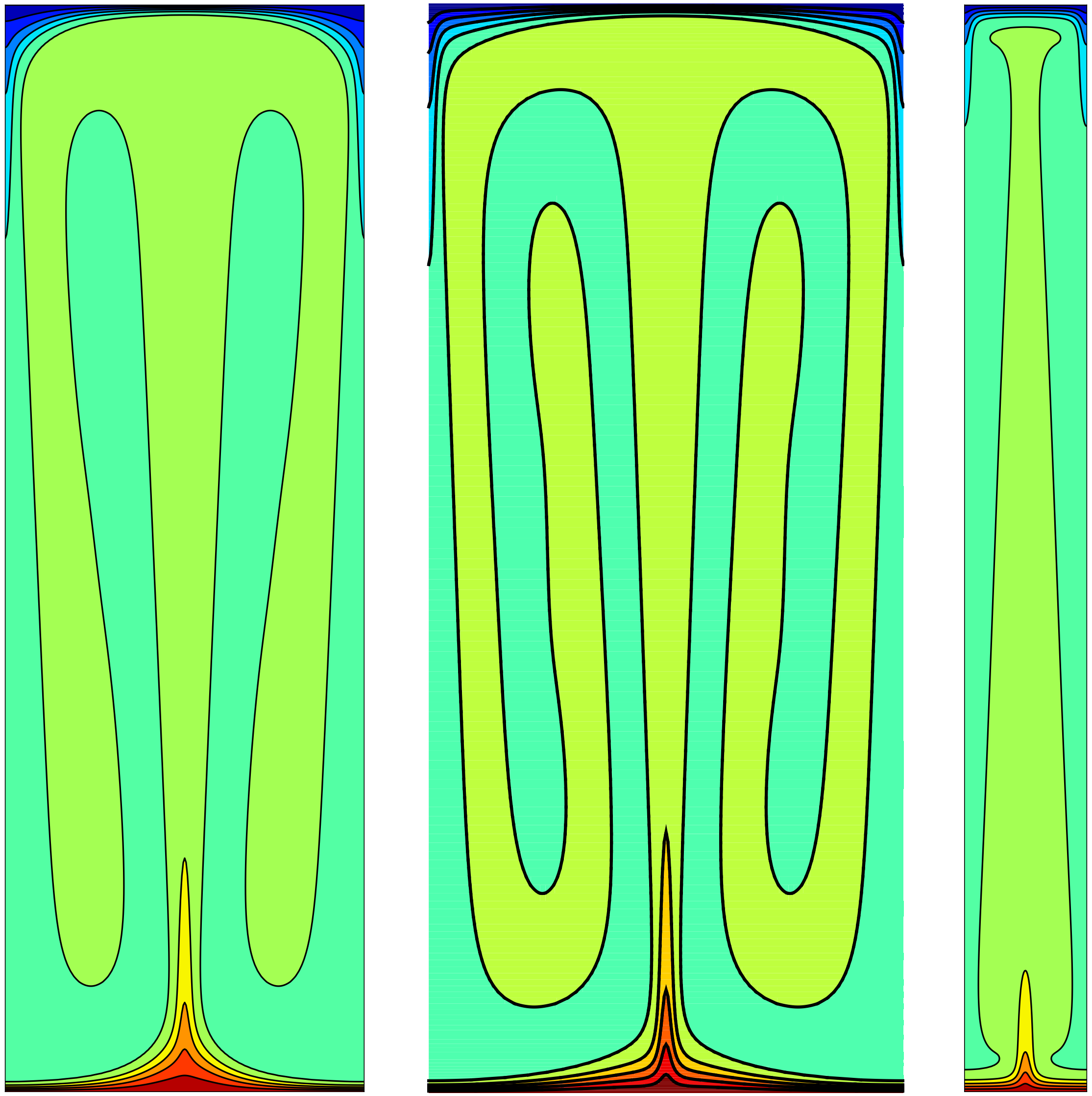
$Nu(\alpha, Ra)$, $Pr = 100$ 1st max disappears



$Nu(\alpha, Ra)$, $Pr = 1, 4, 7, 10, 100$



$Ra = 10^8,\ Pr = 1, 7, 10,$ Temperature



Conclusions: Turbulent transport $\approx$ optimum transport?

- Malkus 1954 theory: maximize Nu, marginal stability
  $\rightarrow Nu \sim Ra^{1/3}$

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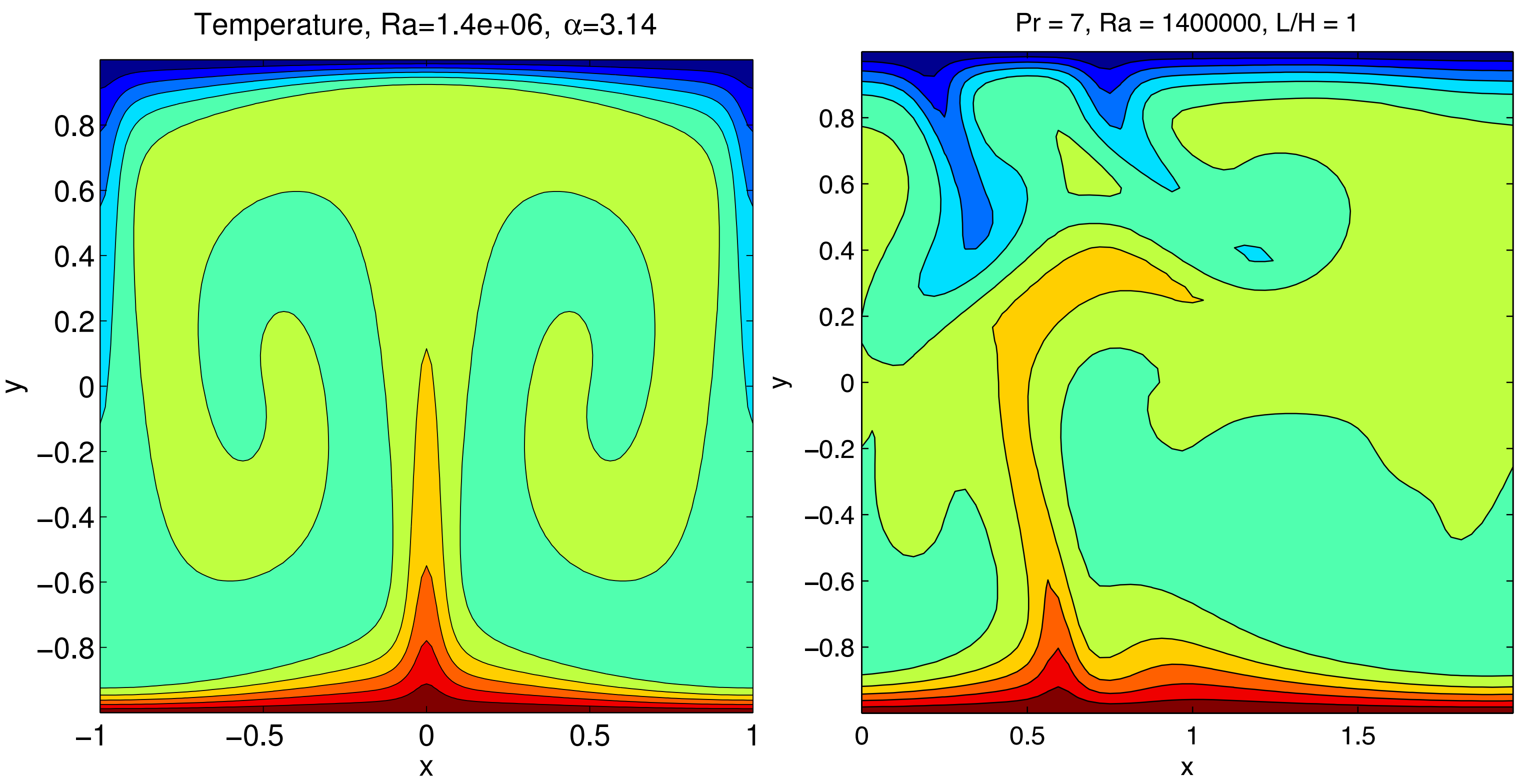
- Whitehead-Doering *2D free-slip* $\rightarrow Nu \lesssim Ra^{5/12}$
- (local?) optimum solutions of full Boussinesq equations
  $\rightarrow Nu \sim Ra^{0.31}$

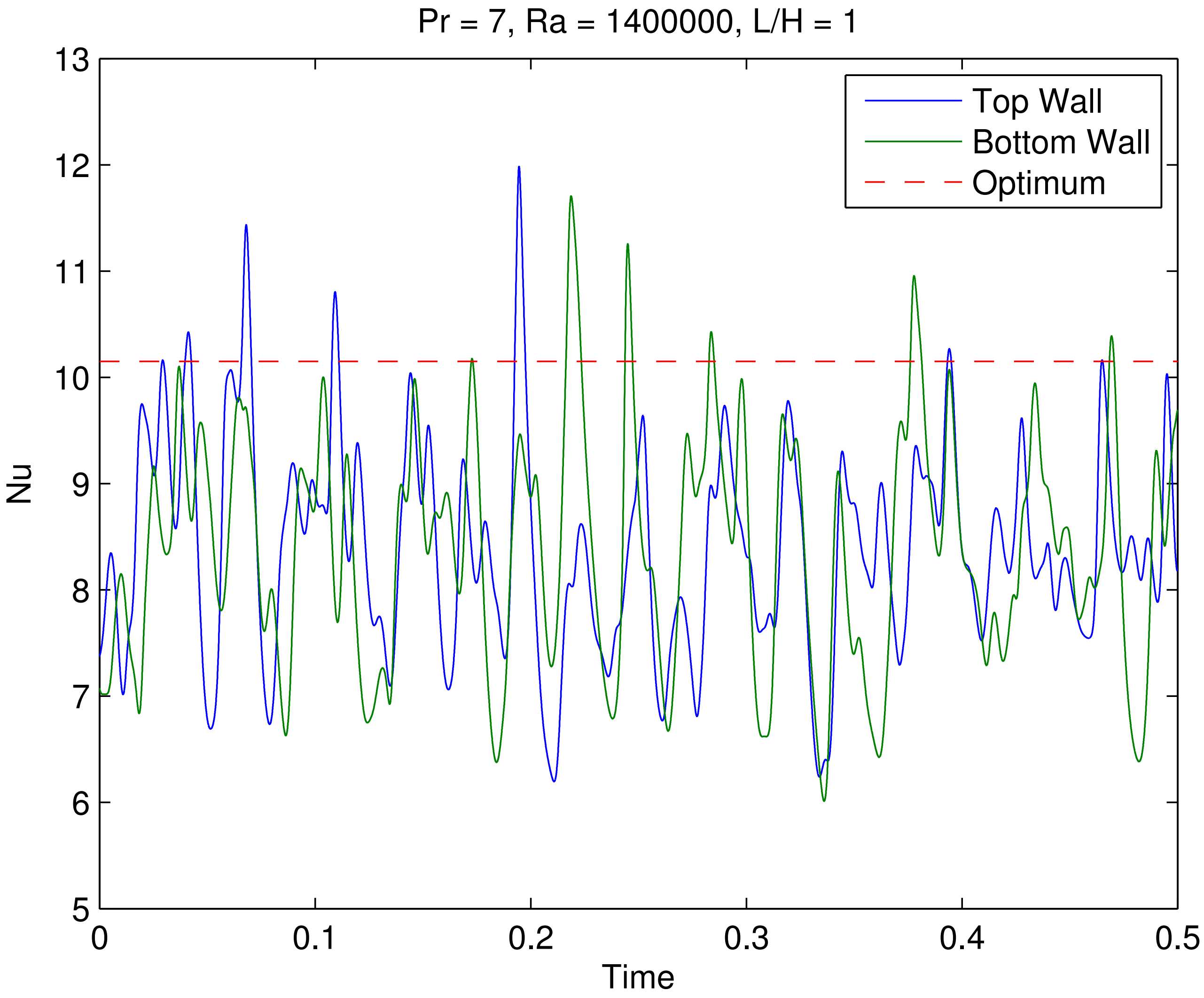
but optimum transport solutions are unstable ?!

- unstable to subharmonics even with mirror symmetry
- unstable to mean shear flow without mirror symmetry
- (weakly) stable with mirror symmetry and no larger scales

Optimum solution: unstable to mean shear flow



Optimum solution: unstable yet tight bound on Nu?



but optimum transport solutions are 2D ?!

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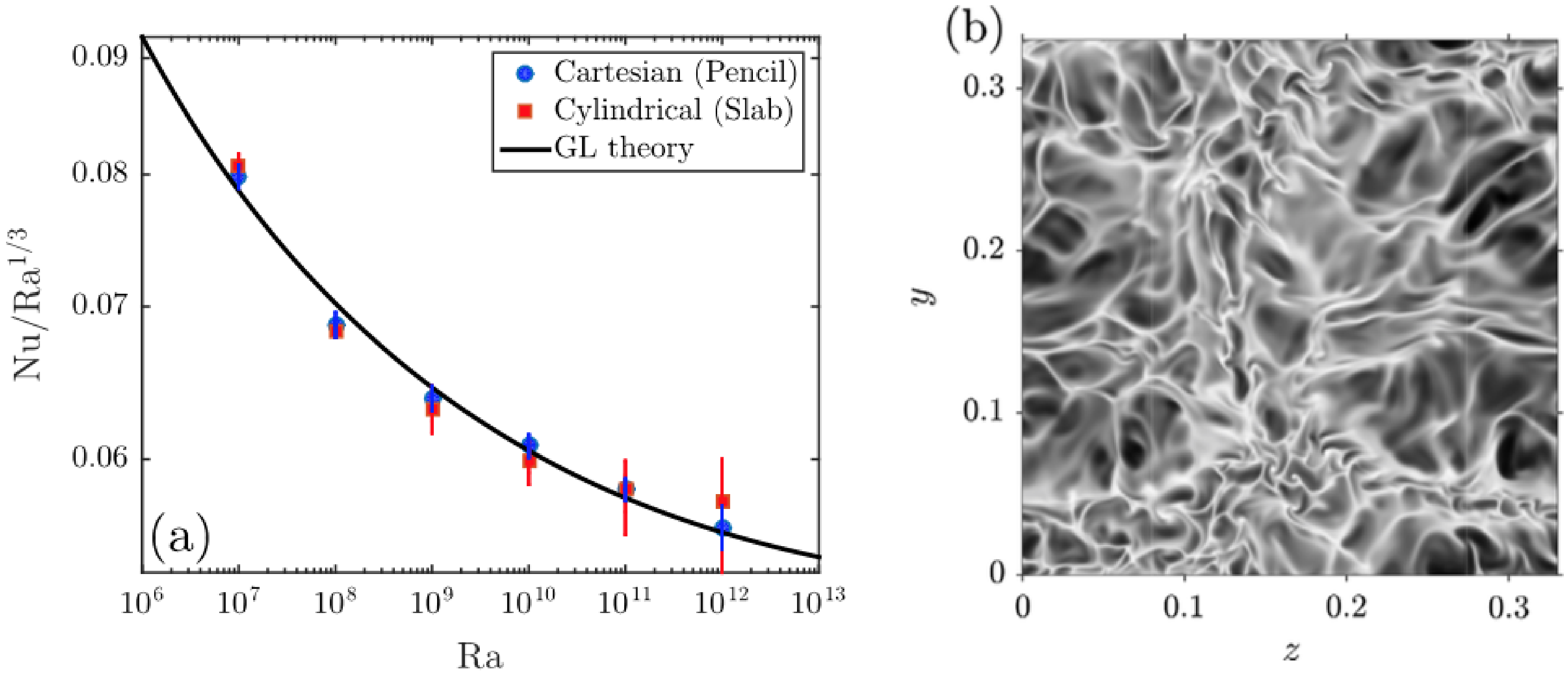
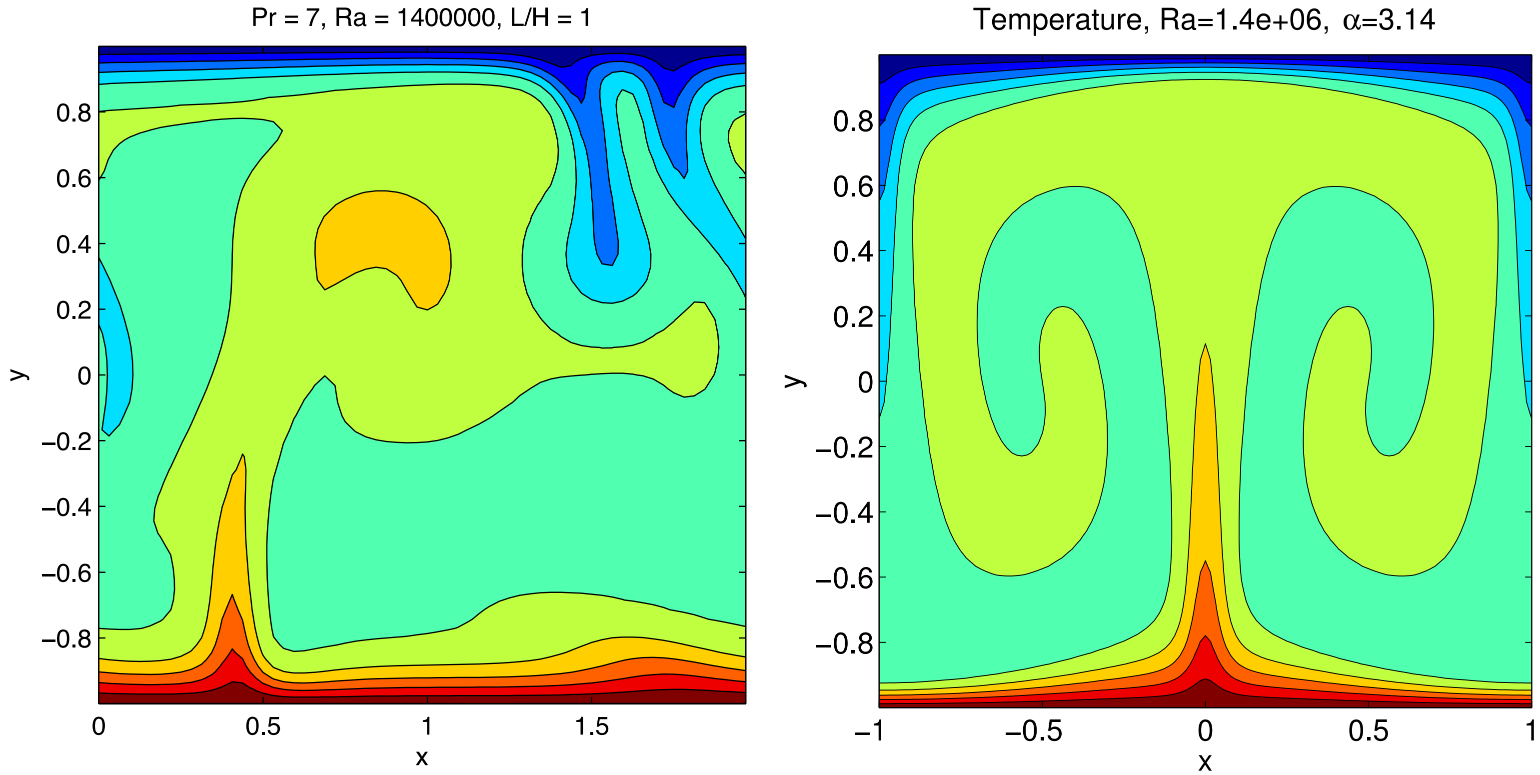
- *really?!*
- yes, possibly

3D Turbulence in RBC: universality and sheets



Cylinder with $D/H = 1/3$ and box (right), $Ra = 10^{11}$, $Pr = 0.7$
van der Poel *et al.*, *Computers & Fluids* 2015

Thank you



Pr = 7, Ra = 1400000, L/H = 1
y
x
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
0.5
1
1.5
Temperature, Ra=1.4e+06, α=3.14
y
x
−1
−0.5
0
0.5
1